MAMA/109, NST3AS/109, MAAS/109

MAT3

MATHEMATICAL TRIPOS Part III

Friday 13 June 2025 1:30 pm to 3:30 pm

PAPER 109

COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove the Kruskal-Katona theorem.

(ii) Deduce that, among families in $[n]^{(r)}$ of given size (where r and n are fixed, with $1 \leqslant r \leqslant n-1$), the size of the upper shadow is minimised by the initial segment of the lexicographic order.

(iii) Is it always true that, among families in $[n]^{(r)}$ of given size, the sum of the sizes of the upper shadow and the lower shadow is minimised by either the initial segment of the colexicographic order or the initial segment of the lexicographic order? Justify your answer.

2

(i) State the Erdős-Ko-Rado theorem, and give two proofs of it: one using the Kruskal-Katona theorem and one using averaging (Katona's method).

(ii) For $1 \leqslant r < n/2$, suppose that A and B are non-empty subsets of $[n]^{(r)}$ such that every set in A meets every set in B. Prove that $|A|, |B|$ cannot both be greater than $\binom{n-1}{r-1}$.

(iii) In (ii), must we always have $|A| + |B| \leqslant 2\binom{n-1}{r-1}$? Justify your answer.

(iv) Use the Frankl-Wilson theorem to prove the following weakening of the Erdős-Ko-Rado theorem: for r fixed and n tending to infinity, the maximum size of an intersecting family in $[n]^{(r)}$ is $\binom{n-1}{r-1}(1 + o(1))$.

3

(i) State and prove the edge-isoperimetric inequality in the discrete cube.

(ii) Is the following statement true or false? For any k, and any $t \leqslant k$, if A is a subset of the two-dimensional grid $[k]^2$ with $|A| = t^2$ then the edge-boundary of A is at least that of the set $[t]^2$. Justify your answer.

(iii) Prove the following statement. For any k, and any $t \leqslant k$, if A is a subset of the two-dimensional grid $[k]^2$ with $|A| = t^2$ then the number of edges spanned by A is at most the number of edges spanned by $[t]^2$. [Hint: Start by down-compressing A.]

(Here as usual we say that an edge xy is *spanned* by A if $x, y \in A$.)

4

(i) State the Frankl-Wilson theorem.

(ii) Show that, for p prime, a family $A \subset [4p]^{(2p)}$ such that $|x \cap y| \neq p$ for all $x, y \in A$ must have size at most $2\binom{4p}{p-1}$.

(iii) State Borsuk's conjecture, and show that it is false.

(iv) Show that there exists a constant C such that, for every n, any bounded set S (consisting of more than 1 point) in n-dimensional Euclidean space may be broken into at most C^n pieces, each of smaller diameter than the diameter of S. [Hint: You may wish to show that a Euclidean ball of radius 1 may be covered by C^n balls of radius 1/4.]

END OF PAPER