UNIVERSITY OF CAMBRIDGE

MAMA/107, NST3AS/107, MAAS/107

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 12 June 2025 1:30 pm to 4:30 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the Dirichlet problem for the Laplacian

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned} \tag{H}$$

for some $\varphi \in C^0(\partial\Omega)$.

(a) Suppose $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ solves (H). State and prove the Mean Value Property and the Weak Maximum Principle for u. Prove moreover that for any multi-index α and $\Omega' \subset\subset \Omega$

$$\|D^\alpha u\|_{L^\infty(\Omega')} \leqslant C\|u\|_{L^1(\Omega)}$$

for some $C = C(n, \alpha, \Omega, \Omega') > 0$.

(b) You can assume that (H) admits a weak solution $u \in C^0(\overline{\Omega})$. Prove that this solution is unique (in $C^0(\overline{\Omega})$) and has the regularity $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$.

(c) By considering a Dirichlet problem for a function v on a ball $B_R(a) \subset\subset \Omega$ with boundary data u, prove the following *local* converse to the Mean Value Property: if $u \in C^0(\Omega)$ satisfies

$$\forall x \in \Omega \ \exists \text{ a sequence of positive numbers } r_j \to 0 \text{ such that}$$

$$u(x) = \frac{1}{|\partial B_{r_j(x)}|} \int_{\partial B_{r_j}(x)} u(y)\, \mathrm{d}y,$$

then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

[Hint: You may use without proof the usual properties of standard mollifiers, provided they are stated clearly. For part (c), you may like to consider the set of points where the difference between v and u is maximised, and argue by contradiction.]

2

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the quasilinear Dirichlet problem

$$\begin{aligned} D_i(a^{ij}(x,u,Du)D_j u)(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= \varphi(x) \quad \text{on } \partial\Omega, \end{aligned} \tag{Q}$$

where $a^{ij}(x,z,\eta) = a^{ji}(x,z,\eta) \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ for some $\alpha \in (0,1)$ and f, $\varphi \in C^\infty(\overline{\Omega})$.

(a) Carefully define what it means for the problem (Q) to be *elliptic*, *strictly elliptic*, and *uniformly elliptic* in Ω.

(b) Let $2 < p < n$ and consider the problem of minimizing the functional

$$F[u] = \frac{1}{p} \int_\Omega |Du|^p \,.$$

Formally derive the Euler–Lagrange equation for $F[u]$ and write down an associated Dirichlet problem in the form (Q). Confirm that $a^{ij}(x,z,\eta) \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ except where $|\eta| = 0$, and show that this problem is elliptic and uniformly elliptic except where $|Du| = 0$, but not strictly elliptic.

(c) By writing down a suitable function space which encodes the boundary data $\varphi \in C^\infty(\overline{\Omega})$, define the notion of a weak solution u to the Dirichlet problem (Q) with $f \equiv 0$ for the operator obtained in (b).

Use the *Direct Method of Calculus of Variations* to prove the existence of such a weak solution, stating clearly the function space to which it belongs.

(d) Consider now the Dirichlet problem (Q) for the operator obtained in (b) with $f \equiv 1$, $\varphi \equiv 0$, and $\Omega = B_1(0)$. Find an explicit solution to this problem. For what values of p is it $C^2(B_1(0))$?

[Hint: You may assume without proof the Poincaré inequality for $W_0^{1,p}(\Omega)$ functions in the form $\|u\|_{L^p(\Omega)} \leqslant C(\Omega,p)\|Du\|_{L^p(\Omega)}$, and the Rellich–Kondrachov embedding $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, where $1 \leqslant q < p^* = \frac{np}{n-p}$. You may also use without proof the fact that the functional $F[u]$ is sequentially weakly lower semicontinuous with respect to convergence in $W^{1,p}(\Omega)$.]

3 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\alpha \in (0,1)$.

(a) Let
$$L = a^{ij}(x)D^2_{ij} + b^i(x)D_i + c(x)$$
be a strictly elliptic operator, where a^{ij}, b^i, $c \in C^{0,\alpha}(\overline{\Omega})$. Suppose that $c \leqslant 0$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu = 0$. State and prove the Weak Maximum Principle for u.

(b) Consider the Dirichlet problem
$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned} \tag{DP}$$
for some $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and $f \in C^{0,\alpha}(\overline{\Omega})$. State, without proof, the global Schauder estimate for u, stating what quantities the constant in the estimate depends on.

(c) Let $\bar{a}^{ij}$, $\bar{b}^i \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, and suppose that the matrix $\bar{a}^{ij}(x,z,\eta)$ is positive-definite for all $(x,z,\eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Consider the quasilinear Dirichlet problem
$$\begin{aligned} Q[u;u] &= 0 \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{QDP}$$
where $Q[u;u] = \bar{a}^{ij}(x,u,Du)D^2_{ij}u + \bar{b}(x,u,Du)$. Define, for $\beta \in (0,1)$, the operator
$$\begin{aligned} T : C^{1,\beta}(\overline{\Omega}) &\to C^{1,\beta}(\overline{\Omega}) \\ v &\mapsto u, \end{aligned}$$
where u solves
$$\begin{aligned} Q[v;u] &= 0 \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{LDP}$$
where $Q[v;u] = \bar{a}^{ij}(x,v,Dv)D^2_{ij}u + \bar{b}(x,v,Dv)$. By proving that the coefficients of the problem (LDP) are suitably regular, show that T is well-defined, and that $T(v) \in C^{2,\alpha\beta}(\overline{\Omega})$.

(d) Prove that T is a compact operator.

(e) Prove that T is a continuous operator.

(f) State the Leray–Schauder Fixed Point Theorem and explain how parts (a)–(e) reduce the existence of a solution to (QDP) to obtaining a uniform estimate for a family of related quasilinear problems, which you should state. Assuming such an estimate holds, explain why the solution to (QDP) has the regularity $C^{2,\alpha}(\overline{\Omega})$. Does this construction also give uniqueness?

[Hint: You may use without proof the fact that a linear problem of the form (DP) *admits a unique $C^{2,\alpha}(\overline{\Omega})$ solution. Recall also that if a sequence has the property that every subsequence has a further subsequence which converges to the same limit, then the whole sequence converges to the same limit.]*

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Throughout this question $L = a^{ij}(x)D^2_{ij} + b^i(x)D_i + c(x)$ is a strictly elliptic operator and $\alpha \in (0,1)$.

(a) Let $l \in \mathbb{N}$. State, without proof, the Hölder interpolation inequality for a function $u \in C^{l,\alpha}(\overline{B_R(x_0)})$.

(b) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that a^{ij}, b^i, $c \in C^{0,\alpha}(\Omega)$, and that $u \in C^{2,\alpha}(\Omega)$ solves $Lu = f \in C^{0,\alpha}(\Omega)$.

State, without proof, the interior Schauder estimate for u.

(c) Now let $B_R^+ = \{(x_1, \ldots, x_{n-1}, x_n) \, : \, x_n > 0\} \cap B_R(0)$, $S_R = \overline{B_R^+} \cap \{x_n = 0\}$, and a^{ij}, b^i, $c \in C^{0,\alpha}(\overline{B_1^+})$, and suppose $u \in C^{2,\alpha}(\overline{B_1^+})$ solves

$$\begin{aligned} Lu = f \in C^{0,\alpha}(\overline{B_1^+}) \quad &\text{in } B_1^+, \\ u = \varphi \in C^{2,\alpha}(\overline{B_1^+}) \quad &\text{on } S_1. \end{aligned}$$

(i) State, without proof, the boundary version of Simon's absorbing lemma.

(ii) State and prove the boundary Schauder estimate for u. *[You may assume in your proof that the right-hand sides of suitable sequences of elliptic problems (which you should obtain) converge to zero locally uniformly. You are **not** required to reproduce identical calculations twice.]*

[Hint: You may additionally use without proof the interior version of Simon's absorbing lemma, the regularity theory for the Laplacian, the reflection principle for harmonic functions, as well as Liouville's theorem for entire harmonic functions, as proven in the lectures.]

END OF PAPER