MAT3

MATHEMATICAL TRIPOS Part III

Monday 9 June 2025 9:00 am to 12:00 pm

PAPER 106

FUNCTIONAL ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let A be a commutative, unital Banach algebra. Define what is meant by a *character* on A. Define the *character space* Φ_A of A. Prove that characters are continuous. [You may assume results about elementary spectral theory of general Banach algebras.]

Show that every maximal ideal of A is closed and is the kernel of a character of A. Deduce that the spectrum of an element $x \in A$ is given by

$$\sigma_A(x) = \{\varphi(x) : \varphi \in \Phi_A\} .$$

Define the *Gelfand topology* of Φ_A and the *Gelfand map* $x \mapsto \hat{x} \colon A \to C(\Phi_A)$. Show that the Gelfand map is a continuous, unital homomorphism.

(b) State the Holomorphic Functional Calculus for an element x of a commutative unital Banach algebra A. Let $f(x)$ be the image of f under the Holomorphic Functional Calculus. Describe how $f(x)$ is defined and use this to prove that

$$\sigma_A(f(x)) = \{f(\lambda) : \lambda \in \sigma_A(x)\} .$$

Let Ω be the unbounded component of $\mathbb{C} \setminus \sigma_A(x)$. Assuming that f is defined on an open neighbourhood of $\mathbb{C} \setminus \Omega$, show that $|f(z)| \leqslant \|f(x)\|$ for all $z \in \mathbb{C} \setminus \Omega$.

(c) Let B be a unital Banach algebra and $x \in B$. For a closed unital subalgebra A of B with $x \in A$, describe topologically how $\sigma_A(x)$ is obtained from $\sigma_B(x)$ in general.

Let A be the closed unital subalgebra of B generated by x. Set $K = \sigma_A(x)$. Prove that $\mathbb{C} \setminus K$ is connected. [Hint: Argue by contradiction. Fix λ in a bounded component of $\mathbb{C} \setminus K$ and approximate the unit $\mathbf{1}$ of A by a polynomial $q(x)$ in x with $q(\lambda) = 0$.]

Show that K is homeomorphic to Φ_A. Identifying K with Φ_A, show that the Gelfand map becomes a map $\theta \colon A \to C(K)$ that satisfies $\theta(f(x)) = f\restriction_K$ for any function f that is holomorphic on some open set containing K. Deduce that any such function f can be uniformly approximated on K by a polynomial.

2

Throughout this question, X is a Banach space whose dual space X^* is separable.

(a) Show that X is separable, B_{X^*} is w^*-metrizable and B_X is w-metrizable.

(b) Show that $x_n \xrightarrow{w} x$ in X if and only if $f(x_n) \to f(x)$ for all $f \in X^*$. Show that every bounded sequence (x_n) in X has a subsequence (y_n) such that $\lim_n f(y_n)$ exists for all $f \in X^*$. Deduce that the difference sequence $(y_{2n} - y_{2n-1})$ is weakly null, i.e., converges weakly to zero.

Suppose that for each $m \in \mathbb{N}$, a weakly null sequence $y_m = (y_{m,n})_n$ in B_X is given. Explain *briefly* why there is a weakly null sequence $z = (z_n)$ in X of which y_m is a subsequence for all $m \in \mathbb{N}$.

(c) Fix a w^*-closed subset K of B_{X^*} and $\varepsilon > 0$. Define

$$K'_\varepsilon = \{f \in K : \text{for all } w^*\text{-neighbourhoods } U \text{ of } f,\ \operatorname{diam}(U \cap K) > \varepsilon\}$$

where $\operatorname{diam} A$ is the $\|\cdot\|$-diameter of a set A defined by $\operatorname{diam} A = \sup\{\|x - y\| : x, y \in A\}$.

(i) Show that K'_ε is a w^*-closed subset of K.

(ii) Show that for each $f \in K'_\varepsilon$ there is a sequence (g_n) in K such that $g_n \xrightarrow{w^*} f$ and $\|g_n - f\| > \varepsilon/2$ for all n.

(iii) Fix $f \in K'_\varepsilon$. Show that there is a sequence (g_n) in K and (x_n) in B_X such that $g_n \xrightarrow{w^*} f$, $|(g_n - f)(x_n)| > \varepsilon/2$ for all n and $|(g_n - f)(x_m)| < \varepsilon/4$ for all m, n with $m < n$. Deduce that there is a sequence (g_n) in K and a weakly null sequence (y_n) in B_X such that $g_n \xrightarrow{w^*} f$ and $|g_n(y_n)| > \varepsilon/16$ for all n.

(iv) Explain why K'_ε is w^*-separable. Let $\{f_m : m \in \mathbb{N}\}$ be w^*-dense in K'_ε. For each $m \in \mathbb{N}$, let $(y_{m,n})_n$ be a weakly null sequence in B_X and $(g_{m,n})_n$ be a sequence in K such that $g_{m,n} \xrightarrow{w^*} f_m$ as $n \to \infty$ and $|g_{m,n}(y_{m,n})| > \varepsilon/16$ for all n. Let (z_n) be a weakly null sequence as in part (b) above. For $N \in \mathbb{N}$ let

$$A_N = \{f \in K : |f(z_n)| \leqslant \varepsilon/20 \text{ for all } n \geqslant N\}\ .$$

Show that if $K \neq \emptyset$, then A_N has non-empty w^*-interior in K for some N. [Hint: Baire category.] Deduce that if $K \neq \emptyset$, then $K'_\varepsilon \subsetneq K$.

3

Denote by $\mathcal{S}_\infty$ the C*-algebra $\mathcal{B}(\ell_2)$ of bounded linear operators on (the complex Hilbert space) ℓ_2 and by (e_n) the standard orthonormal basis of ℓ_2.

(a) Let A be a unital C*-algebra. Define the notion of a *positive* element of A. Prove that a positive element $x \in A$ has a unique positive square root $x^{1/2}$. Show that if $T \in \mathcal{S}_\infty$ is positive, then $\langle Tx, x\rangle \geqslant 0$ for all $x \in \ell_2$.

Let $T \in \mathcal{S}_\infty$. Set $|T| = (T^*T)^{1/2}$. Show that $\ker|T| = \ker T$. We say that $U \in \mathcal{S}_\infty$ is a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in (\ker U)^\perp$. Prove the *polar decomposition for* T: there exists a partial isometry U such that $T = U|T|$ and $\ker U = \ker T$. [Hint: Define U on $\operatorname{im}|T|$ first. Note that $\overline{\operatorname{im}|T|} = (\ker T)^\perp$.]

Let $T \in \mathcal{S}_\infty$ and consider its polar decomposition $T = U|T|$, where U is a partial isometry with $\ker U = \ker T$. Show that $U^*T = |T|$. [Hint: Consider $\langle U^*Tx, y\rangle$ with $y \in \operatorname{im}|T|$ and $y \in \ker T$.]

(b) For $T \in \mathcal{S}_\infty$, set $\|T\|_2 = \Big(\sum_{n=1}^\infty \|Te_n\|^2\Big)^{1/2}$ and $\|T\|_1 = \||T|^{1/2}\|_2^2$. Define $\mathcal{S}_2 = \{T \in \mathcal{S}_\infty : \|T\|_2 < \infty\}$ and $\mathcal{S}_1 = \{T \in \mathcal{S}_\infty : \|T\|_1 < \infty\}$. Prove the following statements for $S, T \in \mathcal{S}_\infty$.

(i) $\|T\|_2 = \|T^*\|_2$ [Hint: Parseval's identity.]

(ii) $\|ST\|_2 \leqslant \|S\|\|T\|_2$ and $\|TS\|_2 \leqslant \|T\|_2\|S\|$ (where $\|\cdot\|$ is the operator norm).

(iii) $\|T\|_1 = \sum_{n=1}^\infty \langle |T|e_n, e_n\rangle$.

(iv) If $T = AB$ with $A, B \in \mathcal{S}_2$, then $T \in \mathcal{S}_1$ and $\|T\|_1 \leqslant \|A\|_2\|B\|_2$. [Hint: Use polar decomposition and (iii).]

(v) If $T \in \mathcal{S}_1$, then there exist $A, B \in \mathcal{S}_2$ such that $T = AB$ and $\|T\|_1 = \|A\|_2\|B\|_2$. [Hint: Use polar decomposition.]

(vi) If $T \in \mathcal{S}_1$, then $ST \in \mathcal{S}_1$ and $\|ST\|_1 \leqslant \|S\|\|T\|_1$. [Hint: Use (v).]

(c) Let $T \in \mathcal{S}_1$. Show that $\sum_{n=1}^\infty \langle Te_n, e_n\rangle$ is absolutely convergent. [Hint: Use (b)(v).] Show that the *trace* $\operatorname{tr}(T)$ of T defined by $\operatorname{tr}(T) = \sum_{n=1}^\infty \langle Te_n, e_n\rangle$ satisfies $|\operatorname{tr}(T)| \leqslant \|T\|_1$.

For $x, y \in \ell_2$ denote by $x \otimes y$ the operator in $\mathcal{S}_\infty$ defined by

$$(x \otimes y)(z) = \langle z, y\rangle x \qquad (z \in \ell_2).$$

Show that $\|x \otimes y\|_1 = \|x\|\|y\|$ and $\operatorname{tr}(x \otimes y) = \langle x, y\rangle$. [Hint: Begin by computing $|x \otimes y|$ for unit vectors x, y. Then use (b)(iii) and Parseval's identities.]

(d) For $S \in \mathcal{S}_\infty$ show that $\|S\| = \sup\{|\operatorname{tr}(ST)| : \|T\|_1 \leqslant 1\}$. [Hint: For one direction, consider $T = x \otimes y$ for unit vectors x, y.]

Assume that $\mathcal{S}_1$ is a normed space with $\|\cdot\|_1$ as its norm. Show that $\mathcal{S}_\infty$ embeds isometrically into the dual space $\mathcal{S}_1^*$. Show that this embedding is surjective, i.e. that $\mathcal{S}_\infty = \mathcal{S}_1^*$, if and only if the space $\mathcal{F}$ of finite-rank operators is dense in $\mathcal{S}_1$. [Hint: In one direction, given $f \in \mathcal{S}_1^*$, consider a sesquilinear form on ℓ_2. You may assume that $\mathcal{F}$ consists of finite sums of operators of the form $x \otimes y$.]

4

(a) Let $(X, \mathcal{P})$ be a real locally convex space. Define the dual space X^* of X.

Let Y be a subspace of X. Show that for all $g \in Y^*$ there exists $f \in X^*$ such that $f\restriction_Y = g$. Show further that if Y is closed and $x_0 \in X \setminus Y$, then there exists $f \in X^*$ such that $f\restriction_Y = 0$ and $f(x_0) \neq 0$. [Characterizations of the continuity of linear functionals on X and the algebraic versions of the Hahn–Banach theorems may be assumed without proof.]

State and prove the Hahn–Banach theorem about the separation of a point and an open convex set in X. [Properties of Minkowski functionals may be used without proof.]

The weak topology $\sigma(X, X^*)$ of X is called the *weak topology* of X. Prove that a closed convex subset K of X is weakly closed. [You may assume without proof any version of the Hahn–Banach separation theorems.] Show that if X is an infinite-dimensional normed space, then the unit sphere S_X is closed but not weakly closed. What is the weak-closure of S_X in X?

(b) Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be locally convex spaces. Let $\mathcal{R}$ be the family of seminorms on $X \times Y$ of the form $(x, y) \mapsto p(x)$ $(p \in \mathcal{P})$ and $(x, y) \mapsto q(y)$ $(q \in \mathcal{Q})$. Verify *briefly* that the topology of the locally convex space $(X \times Y, \mathcal{R})$ is the product topology. Identify, with *brief* justification, the dual space of $X \times Y$.

Let $K_1, \ldots, K_n$ be open convex subsets of X such that $\bigcap_{i=1}^n K_i = \emptyset$. Show that there is a continuous linear map $T\colon X \to \mathbb{R}^{n-1}$ such that $\bigcap_{i=1}^n T(K_i) = \emptyset$. [Hint: consider a suitable open convex set K in X^{n-1} with $0 \notin K$.]

(c) [In answering the questions below, you may use any result from the course. Other results used need to be proved.]

Let $T\colon X \to Y$ be a bounded linear map between Banach spaces. Show that if X is reflexive, then $T(B_X)$ is closed.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V. Let X_j be the normed space $(V, \|\cdot\|_j)$ for $j = 1, 2$. Show that if $X_1^* = X_2^*$, then the two norms are equivalent. [Hint: Consider the closed unit balls B_j of X_j $(j = 1, 2)$.]

END OF PAPER