UNIVERSITY OF CAMBRIDGE

MAMA/105, NST3AS/105, MAAS/105

MAT3

MATHEMATICAL TRIPOS Part III

Monday 16 June 2025 1:30 pm to 4:30 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider the equation

$$\partial_t u = -\partial_x v, \quad \partial_t v = \partial_x u, \quad u(0,x) = f(x), \quad v(0,x) = g(x), \tag{1}$$

on the unknown $(u(t,x), v(t,x))$ valued in $\mathbb{R}^2$ and depending on two variables (t,x) in $\mathbb{R}^2$, with f and g real-analytic near $x = 0$. State the Cauchy-Kovalevskaya theorem for a PDE with a real-valued unknown, and show how it implies that (1) admits a real-analytic solution (u,v) in a neighborhood of $(0,0)$.

(b) Consider the equation (1) again, and assume that f and g are smooth but not real-analytic near $x = 0$. Show that the equation then admits no C^1 solution in any neighborhood of $(0,0)$.

[*Hint: You might want to study the equation satisfied by* $u + iv$.]

(c) Given $\mathcal{U} \subset \mathbb{R}^n$ open and bounded with C^1 boundary, consider the equation

$$-\partial_i\big(a_{ij}\partial_j u\big) = f \ \text{ in } \mathcal{U}, \quad a_{ij}\partial_j u N_i = 0 \ \text{ on } \partial\mathcal{U} \tag{2}$$

with $f \in L^2(\mathcal{U}) \cap C^1(\mathcal{U})$, $a_{ij} = a_{ij}(x)$ measurable and bounded so that $a_{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2$ for some $\theta > 0$ for any $x \in \mathcal{U}$, $\xi \in \mathbb{R}^n$, and $N = N(x)$ normal outgoing vector on $\partial\mathcal{U}$. A classical solution is a function $u \in C^2(\overline{\mathcal{U}})$ that satisfies (2) pointwise. A weak solution is a function $u \in H^1(\mathcal{U})$ that satisfies (denoting by D_i the weak derivative in the direction x_i)

$$\forall\, v \in H^1(\mathcal{U}), \quad \int_{\mathcal{U}} a_{ij} D_i u D_j v = \int_{\mathcal{U}} f v.$$

Prove that $u \in C^2(\overline{\mathcal{U}}) \cap H^1(\mathcal{U})$ is a weak solution if and only if it is a classical solution.

(d) Show that a weak (or classical) solution to (2) exists if and only if $\int_{\mathcal{U}} f = 0$, and comment on the uniqueness of this solution in that circumstance.

[*Hint: You might want to apply the Lax-Milgram theorem in the Hilbert space* $H^1_\dagger(\mathcal{U}) = H^1(\mathcal{U}) \cap \{\int_{\mathcal{U}} u = 0\}$*, which requires proving the Poincaré-Wirtinger inequality* $\|u\|_{L^2(\mathcal{U})} \leqslant C\|Du\|_{L^2(\mathcal{U})}$ *for some* $C > 0$ *in* $H^1_\dagger(\mathcal{U})$*. Approaches based on the Fredholm theory are also acceptable.*]

2

(a) Give the definition of weak (generalised) derivatives and the Sobolev space $W^{1,p}(\mathcal{U})$ for $\mathcal{U} \subset \mathbb{R}^n$ open and $p \in [1, +\infty]$.

(b) Prove that if $u \in W^{1,p}((0,1))$ for some $1 \leqslant p < +\infty$, then u is equal almost everywhere to an absolutely continuous function, which is differentiable almost everywhere with derivative in L^p.

[*Hint: We recall that an absolutely continuous function f is a function so that $f(x) = f(a) + \int_a^x g$ for some Lebesgue integrable g.*]

(c) Given $\mathcal{U} := (0, \infty) \times \mathbb{R}$, prove that for any $u \in W^{1,p}(\mathcal{U})$ there is a sequence $u_j \in C^\infty(\overline{\mathcal{U}})$ so that $u_j \to u$ in $W^{1,p}(\mathcal{U})$.

[*You can assume the existence of a family of standard mollifiers and the smoothness induced by convoluting with such mollifiers, and you can use without proof results about the convergence of the translation in Lebesgue spaces if properly stated.*]

(d) Given $\mathcal{U} := (0, \infty) \times \mathbb{R}$, construct an extension operator E so that for any $u \in W^{1,p}(\mathcal{U})$, $Eu \in W^{1,p}(\mathbb{R}^2)$ with $(Eu)_{|\mathcal{U}} = u$ and $E : W^{1,p}(\mathcal{U}) \to W^{1,p}(\mathbb{R}^2)$ is a linear bounded operator.

[*Hint: You might want to start with $u \in C^\infty(\overline{\mathcal{U}})$ and use the previous question.*]

(e) Given $\mathcal{U} := (0, \infty) \times \mathbb{R}$, construct a trace operator T so that for any $u \in W^{1,p}(\mathcal{U})$, $Tu \in L^p(\mathbb{R})$ with $(Tu)_{|\partial\mathcal{U}} = u_{|\partial\mathcal{U}}$ for any $u \in W^{1,p}(\mathcal{U}) \cap C^\infty(\overline{\mathcal{U}})$, and $T : W^{1,p}(\mathcal{U}) \to L^p(\mathbb{R})$ is a linear bounded operator.

(f) State the Rellich-Kondrachov theorem. Is the unit ball of $W^{1,1}(\mathbb{R}_+)$ compact in $L^1(\mathbb{R}_+)$? (If so, prove it, otherwise provide a counter-example.)

3

Given $F = F(t,x) \in C^1(\mathbb{R}_+ \times \mathbb{R})$ real-valued so that $\partial_x F$ is bounded, $g \in C^1(\mathbb{R}_+ \times \mathbb{R})$ real-valued and $u_0 \in C^1(\mathbb{R})$ real-valued, we consider the following equation

$$\partial_t u + F(t,x)\partial_x u = g, \quad u(0,\cdot) = u_0(\cdot) \tag{1}$$

on the real-valued unknown $u = u(t,x)$ for $t \geqslant 0$ and $x \in \mathbb{R}$.

(a) Define the characteristic trajectories associated with equation (1). Prove that these trajectories exist for all $t \geqslant 0$ and deduce the existence and uniqueness of a classical solution $u \in C^1(\mathbb{R}_+ \times \mathbb{R})$ to the previous equation.

(b) Given $u_0 \in L^\infty(\mathbb{R})$, define a notion of weak solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ to equation (1). Show that, when $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, any such weak solution that is C^1 is in fact the classical solution of the previous question.

(c) When u_0 is merely $L^\infty(\mathbb{R})$ and $g = 0$, prove the existence of a weak solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ by using the characteristic trajectories.

(d) Now consider the nonlinear equation

$$\partial_t u + a\partial_x u = u^2, \quad u(0,x) = \cos x,$$

where $a > 0$ is a constant. Do classical solutions exist for all $t \geqslant 0$? If not provide the first time when the solution becomes infinite.

(e) Adapt your definition of a weak solution in (b) to give a definition of a weak solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ to the quasilinear problem

$$\partial_t u + u\partial_x u = 0, \quad u(0,\cdot) = u_0 \in L^\infty(\mathbb{R}).$$

When $u_0 = 0$, exhibit several weak solutions and therefore prove that weak solutions are not unique.

[*Hint: Prove first that for an a priori piecewise constant solution, on a line of discontinuity* $x = \sigma t$ *with different values* u_- *and* u_+ *left and right of the line, one has* $\sigma = (u_- + u_+)/2$.]

END OF PAPER