UNIVERSITY OF CAMBRIDGE

MAMA/102, NST3AS/102, MAAS/102

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2025 9:00 am to 12:00 pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
Triangular graph paper (3 sheets)

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. In your answers below, you may quote results from the course as long as you clearly state them.

(a) Define the *adjoint representation* and the *Killing form* $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ of $\mathfrak{g}$.

(b) What does it mean for a subset $I \subset \mathfrak{g}$ to be an *ideal* of $\mathfrak{g}$? What does it mean for $\mathfrak{g}$ to be *nilpotent*? Show that if $\mathfrak{g}$ is nilpotent, then κ is identically zero.

(c) Define what it means for $\mathfrak{g}$ to be *solvable*. Let $\mathfrak{g}^{\perp} = \{x \in \mathfrak{g}\colon \kappa(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$. Show that if $\mathfrak{g}$ is solvable, then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}^{\perp}$. Give an example of a solvable $\mathfrak{g}$ whose Killing form is not identically zero.

(d) Show that $\mathfrak{g}^{\perp}$ is a solvable ideal of $\mathfrak{g}$ for every $\mathfrak{g}$.

(e) Define what it means for $\mathfrak{g}$ to be *simple*. Suppose that $\mathfrak{g}$ is simple and let $B\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ be a nondegenerate bilinear form satisfying $B([x, y], z) + B(y, [x, z]) = 0$ for all $x, y, z \in \mathfrak{g}$. Show that there is a nonzero scalar $c \in \mathbb{C}$ such that $\kappa = cB$.

(f) Let E_{ij} be the 3×3 matrix with 1 at position (i, j) and zeroes everywhere else. Compute the determinant of the Killing form of $\mathfrak{sl}_3(\mathbb{C})$ with respect to the standard basis $\{E_{11} - E_{22}, E_{22} - E_{33}, E_{12}, E_{13}, E_{23}, E_{21}, E_{31}, E_{32}\}$.

2 Let E be an ℓ-dimensional Euclidean space and let $\Phi \subset E$ be a root system. In your answers below, you may quote results from the course as long as you clearly state them.

(a) What is a *Weyl chamber*? What does it mean for a subset $\Delta \subset \Phi$ to be a *root basis*? Briefly describe the construction of root bases Δ_γ for certain elements γ (proofs are not required).

(b) Let Δ be a choice of root basis with associated dominant Weyl chamber $\mathcal{C}(\Delta)$. Let W be the Weyl group of Φ. Prove that W acts transitively on the set of root bases of Φ.

(c) Recall that each element $\alpha \in \Delta$ determines a simple reflection $w_\alpha \in W$. Recall also that if $w \in W$, then $\ell(w)$ equals the minimal integer $n \geqslant 0$ with the property that w is a product of n simple reflections. You may use without proof the fact that $\ell(w) = |w(\Phi^+) \cap \Phi^-|$. Suppose that λ, μ lie in the closure of $\mathcal{C}(\Delta)$ and that $w \in W$ has the property that $w(\lambda) = \mu$. Using induction on $\ell(w)$ or otherwise, prove that w equals a product of simple reflections that each fix λ.

(d) Show that for each $\lambda \in E$, the W-orbit of λ contains a unique representative that lies in the closure of $\mathcal{C}(\Delta)$.

3 Let $\mathfrak{g} = \mathfrak{so}_5(\mathbb{C}) = \{X \in \mathfrak{gl}_5(\mathbb{C}) \colon X^t J + JX = 0\}$ where

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{t} \subset \mathfrak{g}$ be the subspace of diagonal matrices. For the first four parts, you do not need to describe proofs for your answers.

(a) Explicitly describe the roots $\Phi \subset \mathfrak{t}^*$.

(b) Find a root basis Δ. Draw the Dynkin diagram of Φ and label it with the elements of Δ.

(c) Let W be the Weyl group of Φ. If $\alpha \in \Phi$, let $w_\alpha \in \Phi$ be the reflection associated to α. Describe the action of w_α on Δ for each $\alpha \in \Delta$.

(d) Write the highest root β as a linear combination of elements of Δ. Compute $\langle \alpha, \check{\beta} \rangle$ for every $\alpha \in \Delta$.

(e) Prove that $\mathfrak{so}_5(\mathbb{C}) \simeq \mathfrak{sp}_4(\mathbb{C})$. You may use results from the course without proof, provided you state them clearly.

4 Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, root system $\Phi \subset \mathfrak{t}^*$ and basis of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Let $\omega_1, \ldots, \omega_\ell$ be the fundamental weights.

(a) If λ is a dominant integral weight, let $V(\lambda)$ be the unique (up to isomorphism) irreducible representation of $\mathfrak{g}$ with highest weight λ. State the Weyl dimension formula for $V(\lambda)$, briefly defining the notation you use.

(b) We say a representation of $\mathfrak{g}$ is *fundamental* if it is isomorphic to $V(\omega_i)$ for some i. Suppose that V is irreducible, nontrivial and $\dim V$ is minimal among the irreducible nontrivial representations of $\mathfrak{g}$. Show that V is fundamental.

(c) Show that every irreducible finite-dimensional representation of $\mathfrak{g}$ is a subrepresentation of some representation of the form $V_1 \otimes \cdots \otimes V_k$, where $k \geqslant 1$ and each V_i is fundamental.

(d) Explicitly describe the fundamental representations for $\mathfrak{sl}_3(\mathbb{C})$.

(e) Let V be the defining representation of $\mathfrak{sl}_3(\mathbb{C})$ and let $\mathfrak{t} \subset \mathfrak{sl}_3(\mathbb{C})$ be the subset of diagonal matrices. Find dominant weights $\lambda_1, \ldots, \lambda_k \in \mathfrak{t}^*$ such that $V \otimes V \otimes V^* \simeq V(\lambda_1) \oplus \cdots \oplus V(\lambda_k)$. [Graphing paper might be useful.]

5 Briefly state (but do not prove) the classification of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, root system $\Phi \subset \mathfrak{t}^*$ and root basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Let V be a finite-dimensional representation of $\mathfrak{g}$. If $\lambda \in \mathfrak{t}^*$, define the λ-*weight space* as $V_\lambda = \{v \in V : t \cdot v = \lambda(t)v \text{ for all } t \in \mathfrak{t}\}$.

(a) Define the *weight lattice* $X \subset \mathfrak{t}^*$. Suppose that $\lambda \in \mathfrak{t}^*$ satisfies $V_\lambda \neq 0$. Show that λ lies in the weight lattice, and show that $V_{w(\lambda)} \neq 0$ for all w in the Weyl group W. [If you wish, you may assume the existence, for each $\alpha \in \Phi$, of a subalgebra $\mathfrak{m}_\alpha = \mathrm{span}\{e_\alpha, h_\alpha, f_\alpha\}$ with $e_\alpha \in \mathfrak{g}_\alpha$, $h_\alpha \in \mathfrak{t}$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ each nonzero and satisfying $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$ and $[e_\alpha, f_\alpha] = h_\alpha$.]

(b) Recall that for $\mu \in \mathfrak{t}^*$ we define the *multiplicity* of μ as $\mathrm{mult}(\mu) = \dim(V_\mu)$. Give an example of a $\mathfrak{g}$, an irreducible V and a $\mu \in \mathfrak{t}^*$ as above with $\mathrm{mult}(\mu) \geqslant 2$.

(c) Define the partial order $\preceq$ on $\mathfrak{t}^*$. Suppose μ, λ are weights of V with μ dominant and with $\mu \preceq \lambda$. Show that $\mathrm{mult}(\mu) \geqslant \mathrm{mult}(\lambda)$.

END OF PAPER