UNIVERSITY OF CAMBRIDGE

MAMA/101, NST3AS/101, MAAS/101

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2025 9:00 am to 12:00 pm

PAPER 101

COMMUTATIVE ALGEBRA

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

The term *ring* stands for a commutative unital ring, and the term *module* stands for a module over such a ring.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let R be a nonzero commutative ring.

i. State the universal property of the tensor product of modules over R.

ii. Suppose A and B are torsion-free abelian groups. Must $A \otimes_{\mathbb{Z}} B$ be a torsion-free abelian group? Give a proof or counter-example as appropriate.

iii. Suppose M is an R-module and $(N_i)_{i=1}^{\infty}$ is a sequence of R-modules. Must the R-modules $M \otimes_R \prod_{i=1}^{\infty} N_i$ and $\prod_{i=1}^{\infty} (M \otimes_R N_i)$ be isomorphic?
Give a proof or counter-example as appropriate.

iv. Prove that the following conditions on R are equivalent:

(a) For all R-modules M and N, if $M \otimes_R N = 0$ then $M = 0$ or $N = 0$.

(b) R is a local ring with maximal ideal $\mathfrak{m}$, and for every R-module M, if $\mathfrak{m}M = M$ then $M = 0$.

v. Let A be an R-algebra, and I, J ideals of R.
Consider the following two statements:

$$I^e \cap J^e \subset (I \cap J)^e \tag{1}$$
$$I^e \cap J^e \supset (I \cap J)^e \tag{2}$$

[$(\cdot)^e$ denotes extension along the ring homomorphism $R \to A$.]

(a) Show that both statements are true if A is flat as an R-module.

(b) For each of the statements, determine whether it is true in general by providing a proof or a counter-example.

2 Let R be a nonzero commutative ring.

i. Define the notion of a multiplicative subset S of R.
State the universal property of the localization of a ring R with respect to a multiplicative subset S.

ii. State and prove the following result: Flatness is a local property of R-modules.

iii. Let $A \subset B$ be an integral extension of rings and $\mathfrak{p} \in \operatorname{spec} A$.

(a) Define the ring $B_{\mathfrak{p}}$, and prove that there is a bijection

$$\{\mathfrak{q} \in \operatorname{spec} B \mid \mathfrak{q} \cap A = \mathfrak{p}\} \leftrightarrow \operatorname{mspec} B_{\mathfrak{p}}$$

given by extension and contraction along the localization map $B \to B_{\mathfrak{p}}$.

(b) Show that the statement of (a) is false in general if the extension $A \subset B$ is not assumed to be integral.

iv. Let I be an ideal of R, and consider the multiplicative set $S = 1+I = \{1+r \mid r \in I\}$. For a maximal ideal $\mathfrak{m}$ of R/I, let $f(\mathfrak{m}) = (\mathfrak{m}^c)^e$, where the contraction is taken along the quotient map $R \to R/I$, and the extension along the localization map $R \to S^{-1}R$. Prove that $f(\mathfrak{m})$ is a maximal ideal of $S^{-1}R$, and that f is a bijection $\operatorname{mspec}(R/I) \to \operatorname{mspec}\left(S^{-1}R\right)$.

3

i. State both the weak and strong forms of the Nullstellensatz, explaining all the notation you use.

Prove the Weak Nullstellensatz.

ii. Prove that every maximal ideal of $\mathbb{Q}[T_1, \ldots, T_n]$ is the contraction of a maximal ideal of $\mathbb{C}[T_1, \ldots, T_n]$.

iii. Consider the ideal $\mathfrak{a} = \left(XY^3, X^2(Y-3)\right)$ of the polynomial ring $\mathbb{R}[X, Y]$. Write down a finite list of polynomials generating the ideal $\sqrt{\mathfrak{a}}$ of $\mathbb{R}[X, Y]$.

iv. Let A be a finitely generated $\mathbb{C}$-algebra. Prove that the cardinality of the set $\hom_{\mathbb{C}}(A, \mathbb{C})$, consisting of all $\mathbb{C}$-algebra homomorphisms $A \to \mathbb{C}$, is not exactly $\aleph_0$.

4

i. Define the following notions:
The height of an ideal (not necessarily prime).
The Krull dimension of a ring.

ii. Let $(A, \mathfrak{m})$ be a noetherian local ring.

(a) Define the associated graded ring $G_{\mathfrak{m}}(A)$.
[**Note:** Define the underlying additive group, and give enough information to deduce how to multiply elements, but do not prove anything.]

Define the number $d(G_{\mathfrak{m}}(A))$.
[**Hint:** This is the order of some pole. You may need to cite a theorem from the lectures to explain the definition.]

State the Dimension Theorem for noetherian local rings, explaining all the notation you use.

(b) For $x \in \mathfrak{m}$, not a zero divisor, prove that $d\left(G_{\mathfrak{m}/(x)}(A/(x))\right) \leqslant d(G_{\mathfrak{m}}(A)) - 1$.

iii. Let $f, g \in \mathbb{C}[X, Y]$ be nonzero polynomials with no common irreducible factor. Prove that the ring $R = \mathbb{C}[X, Y] / (f, g)$ is artinian.

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i. Define the notion of a discrete valuation, and use it to define the notion of a discrete valuation ring.

Define the notion of a Dedekind domain (choose one characterization from the lectures).

ii. Let $(A, \mathfrak{m})$ be an integrally closed noetherian local domain of Krull dimension 1. Prove that $\mathfrak{m}$ is a principal ideal.

iii.

(a) Prove that the localization of a noetherian ring w.r.t. any multiplicative subset is a noetherian ring.

(b) Let A be a Dedekind domain, and $S \subset A$ a multiplicative subset, $0 \notin S$. Prove that $S^{-1}A$ is a Dedekind domain or a field.

END OF PAPER