UNIVERSITY OF CAMBRIDGE

MAMA/358, NST3AS/358, MAAS/358

MAT3
MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 9:00 am to 11:00 am

PAPER 358

INFINITE DIMENSIONAL SPECTRAL COMPUTATION

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define, in terms of the solvability complexity index, the notion of a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$. As an example, give the classical computational spectral problem concerning bounded operators on $l^2(\mathbb{N})$ and give a reason for your choice of the metric space $\mathcal{M}$.

(b) Define a general algorithm. Define the solvability complexity index (SCI) of a problem in this model of computation. Prove that the SCI of the classical computational spectral problem (for the whole class of bounded operators on $l^2(\mathbb{N})$) is at least 3. You may use a suitable decision problem and its SCI discussed in lectures, as long as these results are clearly stated.

(c) Let Ω_{U} be the class of unitary operators on $l^2(\mathbb{N})$. Show that there exists a sequence of arithmetic algorithms $\{\Gamma_n\}_{n\in\mathbb{N}}$ that use the entries of any $A \in \Omega_{\mathrm{U}}$ (with respect to the canonical basis) such that

$$\lim_{n\to\infty} \Gamma_n(A) = \mathrm{Sp}(A) \quad \forall A \in \Omega_{\mathrm{U}}.$$

Is it possible to verify the output $\Gamma_n(A)$ for any given finite n?

2

(a) Give the definition of a projection-valued spectral measure on a separable Hilbert space. State the spectral theorem in terms of projection-valued spectral measures for normal operators A acting on a separable Hilbert space $\mathcal{H}$. Define the scalar-valued spectral measures $\mu_{v,w}$ and μ_v for $v, w \in \mathcal{H}$. Prove that if A is self-adjoint, then μ_v is a positive measure on $\mathbb{R}$ with total mass $\|v\|$.

(b) Now suppose that A is a bounded self-adjoint operator, $\mathcal{H}_n$ are finite-dimensional subspaces of $\mathcal{H}$ and A_n are self-adjoint operators on $\mathcal{H}_n$. Suppose that the orthogonal projections $\mathcal{P}_n$ onto $\mathcal{H}_n$ are such that

$$\lim_{n\to\infty} \mathcal{P}_n^* \mathcal{P}_n v = v \quad \forall v \in \mathcal{H}.$$

What does it mean for the (scalar-valued) spectral measures of A_n to converge weakly to A? Prove that if

$$\lim_{n\to\infty} \langle A_n^m \mathcal{P}_n v, w \rangle = \langle A^m v, w \rangle \quad \forall v, w \in \mathcal{H}, m \in \mathbb{N}, \tag{1}$$

then the spectral measures of A_n converge weakly to A. Does the conclusion still hold if we only require Eq. (1) to hold for $m = 1$?

(c) Now suppose that A is a unitary operator, $\mathcal{H}_n$ are finite-dimensional subspaces of $\mathcal{H}$ and A_n are unitary operators on $\mathcal{H}_n$. Suppose that the orthogonal projections $\mathcal{P}_n$ onto $\mathcal{H}_n$ are such that

$$\lim_{n\to\infty} \mathcal{P}_n^* \mathcal{P}_n v = v \quad \forall v \in \mathcal{H}.$$

What does it mean for A_n to converge to A in the functional calculus sense? Prove that if

$$\lim_{n\to\infty} \langle A_n \mathcal{P}_n v, w \rangle = \langle A v, w \rangle \quad \forall v, w \in \mathcal{H},$$

then A_n converges to A in the functional calculus sense.

(d) Let Ω_{U} denote the class of unitary operators acting on $l^2(\mathbb{N})$. Given a matrix representation of any $A \in \Omega_{\mathrm{U}}$, show how the assumptions in (c) can be realised as an algorithm through a certain polar decomposition of a finite matrix. You may state standard properties of the singular value decomposition of a finite matrix and assume that it can be computed.

3 Throughout this question $(\mathcal{X}, d_{\mathcal{X}})$ is a metric space and ω is a Borel probability measure on $\mathcal{X}$.

(a) Let $F : \mathcal{X} \to \mathcal{X}$ be continuous. What does it mean for

(i) F to be non-singular with respect to ω?

(ii) F to be measure-preserving with respect to ω?

(ii) F to be invertible with respect to ω?

For case (i), define the Koopman operator K_F on $L^\infty(\mathcal{X}, \omega)$ and state a necessary and sufficient condition for it to extend to a bounded operator on $L^2(\mathcal{X}, \omega)$.

(b) If $(X, \omega; F)$ is an invertible measure-preserving system, we say that the system is *ergodic* if every invariant set has measure 0 or 1.

(i) Show that an invertible measure-preserving system is ergodic if and only if whenever $K_F g = g$ for $g \in L^2(\mathcal{X}, \omega)$, g is constant ω-almost everywhere.

(ii) For $a \in (0, 2\pi)$, consider the circle rotation

$$X = [-\pi, \pi]_{\text{per}}, \quad F(x) = x + a,$$

where ω is the normalised Lebesgue measure. Prove that the system is ergodic if and only if a/π is irrational.

(c) Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, define the computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}^{\Delta_1}$ with inexact information. Suppose that we fix the metric space $(\mathcal{X}, d_{\mathcal{X}})$ and measure ω. Let Ω be a class of continuous maps $F : \mathcal{X} \to \mathcal{X}$ that are non-singular with respect to ω. Without specifying Ξ and $\mathcal{M}$, provide a suitable Λ so that $\{\Xi, \Omega, \mathcal{M}, \Lambda\}^{\Delta_1}$ corresponds to a "perfect measurement device" and explain this terminology.

(d) Again let $X = [-\pi, \pi]_{\text{per}}$, ω the normalised Lebesgue measure and

$$\Omega = \{F : \mathcal{X} \to \mathcal{X} \text{ such that } F \text{ is continuous, measure-preserving and invertible}\}.$$

Consider the problem function

$$\Xi_{\text{erg}} : \Omega \to \{0,1\} \text{ (with the discrete topology) } ; F \mapsto \begin{cases} 1, & \text{if } F \text{ is ergodic,} \\ 0, & \text{otherwise.} \end{cases}$$

With your choice of Λ in (c) prove that $\{\Xi_{\text{erg}}, \Omega, \{0,1\}, \Lambda\}^{\Delta_1} \notin \Delta_2^G$.
(Recall that $\notin \Delta_2^G$ means the SCI of the problem is at least two.)

END OF PAPER