MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 11 June 2024 1:30 pm to 3:30 pm

PAPER 357

GRAVITATIONAL WAVES AND NUMERICAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In coordinates adapted to the 3+1 split of the Einstein equations, the line element is given by

$$\mathrm{d}s^2 = (-\alpha^2 + \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\,\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\,\mathrm{d}x^j\,,$$

where $\alpha,\ \beta^i,\ \gamma_{ij}$ denote the lapse function, shift vector and spatial metric, respectively, Latin indices run from 1 to 3 and repeated indices are summed over. The timelike unit normal and extrinsic curvature are defined by

$$n^\mu = \left(\frac{1}{\alpha},\ -\frac{\beta^i}{\alpha}\right)\,, \qquad K_{\alpha\beta} = -\perp^\mu{}_\alpha \nabla_\mu n_\beta\,,$$

with the projector $\perp^\mu{}_\alpha = \delta^\mu{}_\alpha + n^\mu n_\alpha$. The three-dimensional covariant derivative of a tensor $T^{\alpha\ldots}{}_{\beta\ldots}$ is defined in terms of its spacetime covariant derivative by

$$D_\mu T^{\alpha\ldots}{}_{\beta\ldots} = \perp^\rho{}_\mu \perp^\alpha{}_\sigma \perp^\tau{}_\beta \ldots \nabla_\rho T^{\sigma\ldots}{}_{\tau\ldots}\,.$$

Let Z^μ be a vector field with time and spatial projections

$$\Theta = -n_\mu Z^\mu\,, \qquad \Theta_\mu = \perp^\rho{}_\mu Z_\rho\,.$$

(a) Show that

(i) $\quad Z_\mu = \Theta_\mu + n_\mu \Theta\,,$

(ii) $\quad -\nabla_\mu n_\nu = K_{\mu\nu} + n_\mu a_\nu\,, \quad$ where $\quad a_\nu = n^\mu \nabla_\mu n_\nu\,,$

(iii) $\quad \nabla^\mu Z_\mu = -n^\mu n^\nu \nabla_\mu Z_\nu + D^\mu Z_\mu\,,$

(iv) $\quad D^\mu Z_\mu = D^\mu \Theta_\mu - K\Theta\,, \quad$ where $\quad K = K^\mu{}_\mu\,.$

(b) Now consider the generalized Einstein equations

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu - [n_\mu Z_\nu + n_\nu Z_\mu - g_{\mu\nu} n_\sigma Z^\sigma] = 8\pi\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right)\,.$$

Using the scalar Gauss equation,

$$R^\mu{}_\mu + 2R_{\mu\nu} n^\mu n^\nu = \mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu}\,,$$

show that

$$\mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu} - 2n^\mu \nabla_\mu \Theta - 2Z_\mu a^\mu - 4\Theta + 2D^\mu Z_\mu = 16\pi\rho\,.$$

Here $\rho = T_{\mu\nu} n^\mu n^\nu$ and $\mathcal{R}$ denotes the Ricci scalar associated with the spatial metric γ_{ij}.

By expanding $n^\mu \nabla_\mu \Theta$, show that the Θ obeys the time evolution equation

$$\partial_t \Theta \;=\; \beta^m \partial_m \Theta + \frac{\alpha}{2}\big[\mathcal{R} + K(K + d_1\Theta) - K_{\mu\nu}K^{\mu\nu} + d_2 \Theta_\mu a^\mu - 4\Theta + d_3 D^\mu \Theta_\mu + d_4 \rho\big]\,,$$

where $d_1,\ d_2,\ d_3,\ d_4$ are constants you should determine.

2

In natural units, where $G = c = 1$, the Schwarzschild metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{R}\right)\mathrm{d}T^2 + \left(1 - \frac{2M}{R}\right)^{-1}\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2 \quad \text{with} \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,.$$

(a) Compute the Schwarzschild metric in the coordinates (t, R, θ, ϕ) where the new time t is defined by

$$\mathrm{d}T = \mathrm{d}t - \frac{RM}{(R-M)(R-2M)}\mathrm{d}R\,.$$

(b) Introducing also a new radial coordinate $r = R - M$, show that the Schwarzschild metric can be written as

$$\mathrm{d}s^2 = -\frac{r-M}{r+M}\mathrm{d}t^2 + \frac{2M}{r}\mathrm{d}t\,\mathrm{d}r + \psi^4(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) \quad \text{where} \quad \psi^4 = \left(\frac{r+M}{r}\right)^2. \qquad (\dagger)$$

(c) By comparing the metric (†) with the 3+1 metric

$$\mathrm{d}s^2 = (-\alpha^2 + \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i\mathrm{d}t\,\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\,\mathrm{d}x^j\,,$$

where Latin indices run over (r, θ, ϕ) and repeated indices are summed over, determine the lapse function α (assuming $\alpha > 0$), the shift vector components β^i (note the upstairs index) and the spatial metric γ_{ij}.

(d) Use the evolution equation

$$\partial_t\gamma_{ij} = \beta^m\partial_m\gamma_{ij} + \gamma_{mi}\partial_j\beta^m + \gamma_{mj}\partial_i\beta^m - 2\alpha K_{ij}\,,$$

to compute the non-vanishing components of the extrinsic curvature K_{ij}. Show that the trace $K = \gamma^{ij}K_{ij}$ of the extrinsic curvature is $K = \frac{M}{(r+M)^2}$.

(e) Consider the Bona-Massó family of slicing conditions,

$$(\partial_t - \beta^i\partial_i)\alpha = -\alpha^2 f(\alpha)K\,,$$

and determine the function $f(\alpha)$ such that this slicing condition is satisfied by the Schwarzschild metric in the form (†). Deduce that $f(\alpha)$ vanishes in the limit $\alpha \to 1$.

3

(a) Consider the Minkowski metric in Cartesian coordinates,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,.$$

Compute this line element in characteristic coordinates defined by $u = t-z$, $v = t+z$ with x and y unchanged. Graphically sketch the coordinate basis vectors ∂_t, ∂_z, ∂_u, ∂_v in the plane $x = \text{const}$, $y = \text{const}$ with the t direction pointing upwards and the z direction pointing horizontally. Express the wave operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ in the characteristic coordinates (u, v, x, y).

(b) In linearized theory, the metric is given by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $h_{\alpha\beta} \ll 1$. A plane wave solution to the linearized Einstein equations $\Box h_{\mu\nu}$ in transverse traceless gauge with background coordinates (t, x, y, z) is given by

$$h_{\mu\nu} = H_{\mu\nu} e^{\mathrm{i}p_\rho x^\rho} \qquad \text{with} \qquad H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad p_\sigma = \omega(-1,\, 0,\, 0,\, 1)\,,$$

where H_+, $H_\times$ and ω are real constants. Starting from its definition $\Psi_4 = R_{\mu\nu\rho\sigma}\mathsf{k}^\mu \bar{\mathsf{m}}^\nu \mathsf{k}^\rho \bar{\mathsf{m}}^\sigma$, compute the Newman-Penrose scalar Ψ_4 at linear order for this plane wave. Here, the linearized Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho})\,,$$

and the tetrad vectors are defined in terms of the coordinate basis vectors by

$$\mathsf{k} = \tfrac{1}{\sqrt{2}}(\partial_t - \partial_z)\,, \qquad \bar{\mathsf{m}} = \tfrac{1}{\sqrt{2}}(\partial_x - \mathrm{i}\partial_y)\,.$$

(c) Express the plane wave solution of part (b) in terms of the characteristic coordinates (u, v, x, y) and explicitly show that this solution satisfies the linearized Einstein equations $\Box h_{\mu\nu} = 0$ with the wave operator in characteristic coordinates as determined in part (a).

(d) Consider the spacetime metric

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + f(u)^2\mathrm{d}x^2 + g(u)^2\mathrm{d}y^2.$$

Denoting $f' = \mathrm{d}f/\mathrm{d}u$ and $g' = \mathrm{d}g/\mathrm{d}u$, the only non-vanishing Christoffel symbols for this metric are

$$\Gamma^x_{xu} = \frac{f'}{f}\,, \quad \Gamma^y_{yu} = \frac{g'}{g}\,, \quad \Gamma^v_{xx} = 2ff'\,, \quad \Gamma^v_{yy} = 2gg'\,,$$

and those related to these by symmetry. In the following you may assume without proof that $R_{uu} = R^\alpha{}_{u\alpha u}$, is the only component of the Ricci tensor that does not trivially vanish. Determine the vacuum Einstein equation $R_{uu} = 0$ for this spacetime by computing all necessary components of the Riemann tensor

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\delta} - \partial_\delta\Gamma^\alpha_{\beta\gamma} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta}\,.$$

Consider the plane wave of part (c) with $H_\times = 0$. Determine how H_+ of this plane wave is related to $f(u)$ and $g(u)$ and show that the plane wave is a solution to the Einstein equations at linear order in H_+.

END OF PAPER