MAMA/355, NST3AS/355, MAAS/355

MAT3
MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 9:00 am to 12:00 pm

PAPER 355

BIOLOGICAL PHYSICS AND FLUID DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

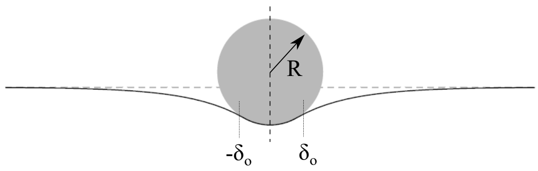
Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 An elastic membrane with bending modulus k_c and tension σ initially lies flat in the $x-y$ plane. An infinite cylinder of radius R is then brought into contact with the membrane, deforming it into the shape shown in cross section in the figure.



Assume the limit of small membrane deformations $h(x)$, where, relative to the flat configuration, the membrane energy E_m per unit length along the cylinder is

$$E_m = \frac{1}{2}\int \left(k_c h_{xx}^2 + \sigma h_x^2\right) dx,$$

and that the membrane adheres to the cylinder in the region $-\delta_o \leqslant x \leqslant \delta_o$, where the centre of the cylinder is at $x=0$, with $\delta_o \ll R$ and also $\delta_o \ll \xi$, where $\xi = \sqrt{k_c/\sigma}$. The adhesion energy E_a of the membrane to the cylinder is $-\mathcal{U}$ per unit area.

(a) Find the Euler-Lagrange equation that governs the membrane shape in regions $|x| \geqslant \delta_o$, and find its general solution. Show that the elastic energy in the deformed membrane outside the contact region can be expressed entirely in terms of the function h and its derivatives at the contact points $x = \pm\delta_o$. By enforcing suitable matching conditions at the contact points, find the shape of the free parts of the membrane and thus their elastic energy. By minimising the total energy $E = E_m + E_a$ with respect to δ_o, show that the minimum energy configuration has

$$\delta_o = \xi\left(U - \frac{1}{2}\right),$$

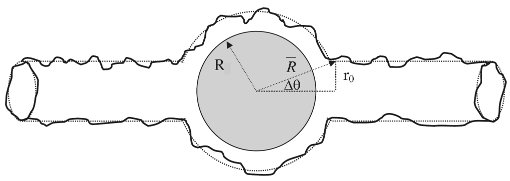
where $U = R^2\mathcal{U}/k_c$, and thus binding is only favorable for sufficiently large U. Show that when the membrane is bound to the cylinder the configuration has energy

$$E = -\frac{k_c\xi}{R^2}\left(U - \frac{1}{2}\right)^2.$$

(b) Now consider two such cylinders whose centres are at $x = \pm L$ and which adhere to the same side of the membrane, with $L > R$. Repeating the analysis in (a), but allowing for distinct outer and inner contact lengths δ_o and δ_i for the membrane outside the two cylinders and in the region in between, show that there is a repulsive interaction between the cylinders with energy

$$V(L) = \frac{k_c\xi}{R^2}\left(U - \frac{1}{2}\right)^2\left[1 - \tanh\left(\frac{L}{\xi}\right)\right].$$

2 Certain cells in the retina can be modelled as cylindrical membranes of bending modulus k_c that enclose a spherical nucleus of radius R as in the figure.



The membrane energy is the sum of bending elasticity and an area term with a tension σ,

$$E_m = \int dS \left(\sigma + \frac{1}{2} k_c (2H)^2 \right).$$

where H is the mean curvature.

(a) Show that the equilibrium radius r_0 of the cylinder far from the sphere is $r_0 = \sqrt{k_c/2\sigma}$.

(b) Entropic repulsion from membrane fluctuations increases the average radius of the membrane to $\bar{R} > R$. Using Helfrich's form for the free energy $F(d) = (k_B T)^2/64 k_c d^2$ per unit area for a membrane a distance d from a wall, write down an approximate expression for the total free energy of this system as a function of $\delta = \bar{R} - R$, in the high-tension regime where $r_0 \ll R$ and $\delta/R \ll 1$. Minimizing with respect to δ, show that

$$\delta = \left(\frac{(k_B T)^2 R}{64 k_c \sigma} \right)^{1/3}.$$

(c) The sphere now moves along the tube at speed U. In the frame of the sphere, the fluid in the thin gap can be described by lubrication theory with a separable azimuthal velocity $u_\theta = \phi(z)/\sin\theta$, where $z \in [0, \delta]$ is a radial coordinate within the gap. Show that the Stokes equation coupling the pressure p to ϕ becomes

$$\sin\theta \frac{\partial p}{\partial \theta} = \mu R \frac{\partial^2 \phi}{\partial z^2} = a,$$

for some constant a, and therefore

$$p(\theta) = \frac{1}{2} a \ln \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right).$$

In the frame moving with the sphere, find $\phi(z)$ by enforcing fluid conservation in the limit $r_0/R \ll 1$ and the boundary conditions at $z = 0$ and $z = \delta$. Show by a scaling argument that the pressure contribution to the force on the sphere dominates the viscous one, and compute the total force on the sphere as

$$2\pi R^2 \int_{\Delta\theta}^{\pi - \Delta\theta} d\theta \sin\theta \cos\theta \, p(\theta).$$

In the limit $\Delta\theta \ll 1$, show that the drag coefficient on the sphere is enhanced from its value $6\pi\mu R$ in the absence of the membrane by the factor $2R^2/\delta^2$. How does the diffusion constant of the sphere change?

$\left[Hint : \int_{-c}^{c} x \ln \left(\frac{1+x}{1-x} \right) dx = 2c + (1 - c^2) \ln \left(\frac{1-c}{1+c} \right). \right]$

3 A spherical cell swims a distance h from a flat free surface and is tilted at an angle θ from it, where $\theta = 0$ means the cell is swimming parallel to the surface. The flow created by a cell away from the surface is that of an axisymmetric force dipole of magnitude $\mathcal{P}$ and direction $\mathbf{p}$,

$$\mathbf{u}(\mathbf{r}) = \frac{\mathcal{P}}{8\pi\mu}\left(\frac{3\,(\mathbf{p}\cdot\mathbf{r})^2}{r^5} - \frac{1}{r^3}\right)\mathbf{r}.$$

(a) Show that all boundary conditions on the free surface (no shear and no penetration) are satisfied by the superposition of the original dipole and a mirror-image dipole located on the other side of the surface at a distance h away. Deduce the value of the perturbation flow induced by the free surface on the cell and hence the surface-induced motion of the cell. Explain the different behaviour in the cases of pullers and pushers.

(b) The orientation of the cell in an external flow whose vorticity is $\boldsymbol{\omega}$ may be described using the general equation

$$\dot{\mathbf{p}} = \boldsymbol{\Omega}_f \times \mathbf{p}, \qquad \boldsymbol{\Omega}_f = \frac{1}{2}\boldsymbol{\omega}.$$

Find the equation of motion of the orientation angle θ. Using this result, explain the different behaviours expected for pullers and pushers near a free surface.

END OF PAPER