UNIVERSITY OF CAMBRIDGE

MAMA/349, NST3AS/349, MAAS/349

MAT3

MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 9:00 am to 12:00 pm

PAPER 349

THE LIFE AND DEATH OF GALAXIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Describe the Strömgren sphere model and derive the radius of the Strömgren sphere for a cloud of hydrogen gas with a number density n and a recombination coefficient α around a photon source with luminosity L_ν.

Obtain the equation governing the radius of a dusty Strömgren sphere by including absorption by dust grains with cross-section σ_{d} and dust-to-gas fraction f_{d}. Start with a cloud of arbitrary number density $n(r)$ and assume that the optical depth is dominated by the dust absorption with only a small contribution due to photoionization processes. It is useful to think of the action of dust in terms of the volume density of the energy absorption rate (by analogy with volume density of recombination rate).

Finally simplify the equation for the radius of the dusty Strömgren sphere assuming the gas density is constant.

2

Describe the metallicity-stellar mass relationship in galaxies. Pay attention to the observable quantities involved: what, where and how is measured to be included in this picture. What is the main hypothesized cause of the change of the metallicity with mass? Inspired by this hypothesis, what changes to the Closed Box chemical enrichment model are introduced?

Derive the equation connecting the galaxy's gas-phase metallicity, $Z_g = M_{z,g}/M_g$ and its stellar metallicity, $Z_* = M_{z,*}/M_*$. If both stellar and gas-phase metallicities are measured, show how the mass-loading factor λ can be estimated. Hint: write down the equation for the evolution of mass of metals in the galaxy's gas in the presence of outflows. Re-write the equation in terms of the change in stellar mass, $\dot{M}_*$ and note that the change in the mass of metals locked in stars is simply $\dot{M}_{z,*} = Z_g \dot{M}_*$. For convenience, you might want to introduce gas-to-stellar mass fraction $r_g = M_g/M_*$

In the Milky Way, the failure of the Closed Box model is often illustrated with the G-dwarf problem. State the problem, paying attention to the data and the model interpretations. Perhaps the G-dwarf problem can be solved using metallicity-dependent yields? Explain why introducing the increase in yield with growing metallicity may seem appealing as a potential solution of the G-dwarf problem. Test if such a modification of the Closed Box model actually works by introducing yield linearly dependent on metallicity, i.e. $y_z = kZ$, and computing the corresponding Z as a function of M_g and $M_*(< Z)$ as a function of Z.

3

One of the most powerful and versatile tools in Galactic Dynamics is the Distribution Function. Give the definition of the Distribution Function and describe how it is used to model observed properties of galaxies. The exact functional form for Galactic DFs is clarified by the Jeans Theorem. Give a two-sentence formulation of the Jeans Theorem.

Although DF is conceptually clear and highly effective, not many useful functional forms are known. Here is a simple power-law DF which depends on position and velocity only through energy, and which can be shown to have several handy analytic properties:

$$f = \begin{cases} F\mathcal{E}^{n-\frac{3}{2}}, & \mathcal{E} > 0 \\ 0, & \mathcal{E} \leqslant 0 \end{cases}$$

where F is a constant, $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$ is relative energy and $\Psi = \Phi + \Phi_0$ is relative potential. Using the substitution $v^2 = 2\Psi\cos^2\theta$ show that the DF's corresponding density

$$\rho \propto \Psi^n$$

and derive the dependence of the velocity dispersion on the relative potential

$$\overline{v^2} \propto \Psi$$

Hint: when computing the second moment of the velocity distribution, do not forget to divide by the normalising integral of the DF.

4

What is dynamical friction and what role does it play in galaxy evolution? Write down the dynamical friction force acting on a body of mass M moving with a velocity v in a medium with density ρ. Hint: If you do not remember the force formula, it can be derived assuming that in the reference frame of the massive body, particles with a critical impact parameter $b_{\rm cr}$ lose all their angular momentum to M. The critical impact parameter can be obtained assuming that the change in the perpendicular velocity component of the particle, equal to the force at the closest approach multiplied by the time spent there, is of order of v.

For a satellite of mass M_s on a circular orbit in a host galaxy with a flat rotation curve v_c = const, compute the time to sink from the initial host-centric distance r_0 to the centre of the host. Hint: assume that the satellite's orbit remains circular and that the rate of change in the angular momentum of the in-falling satellite is provided by the torque exerted by the dynamical friction. Interpret the result obtained.

Finally, re-derive the sinking time taking into account that the satellite will lose mass to the tidal force of the host. Assume that the satellite's density also gives rise to a flat rotation curve and its initial mass at r_0 is M_s and the size is limited to the tidal radis. To simplify the calculation of the tidal radius $R_t(r)$ of the satellite, assume that this limiting radius is the distance from the satellite's centre where its density is equal to the density of the host $\rho_h(r)$. Compare to the earlier estimate and interpret the result obtained.

END OF PAPER