MAT3

MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 1:30 pm to 3:30 pm

PAPER 347

ASTROPHYSICAL BLACK HOLES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider a spherically symmetric gas cloud with a constant density ρ_0, and a radius r. If the condition for gravitational collapse is satisfied, calculate the Jeans mass of this cloud $M_{\rm J}$, as a function of ρ_0 and cloud temperature T, where you may assume that the adiabatic index is $\gamma = 5/3$.

(b) Using the results from (a), determine how $M_{\rm J}$ scales with gas density in the case of both adiabatic and isothermal collapse and hence, explain how these findings help us interpret the formation and initial mass function of PopIII and PopII stars.

Futhermore, what can we infer about the black hole seed formation mechanisms from these considerations?

(c) Consider now an *initially* axisymmetric, infinitesimally thin disc with constant surface density Σ_0, constant gas pressure p_0, and angular velocity $\Omega(R)$, subject to its own self-gravity with gravitational potential Φ_0. Assume that small axisymmetric perturbations in the disc lead to the following changes in the disc properties:

$$\Sigma = \Sigma_0 + \Sigma' \,, \quad p = p_0 + p' \,, \quad u_R = u'_R \,, \quad u_\phi = R\Omega + u'_\phi \,, \quad \Phi = \Phi_0 + \Phi' \,,$$

where perturbed quantities are indicated with a superscript $'$, u_R is the radial velocity, u_ϕ is the azimuthal velocity and R is the cylindrical radius. Write down the mass conservation, momentum and Poisson equations both for the unperturbed and perturbed disc.

Assuming that the perturbations vary as $\propto e^{i(kR-\omega t)}$ analyse these equations in the Fourier space, where you may retain only first order perturbation terms, to obtain the following dispersion relation:

$$\omega^2 = \kappa^2 - 2\pi G|k|\Sigma_0 + k^2 c_{\rm s}^2 \,. \tag{1}$$

Here k is the radial wavenumber, ω is the angular frequency, $\kappa^2 \equiv 4\Omega^2 + 2R\Omega\frac{d\Omega}{dR}$ is the epicyclic frequency, and $c_{\rm s} = \left(\frac{dp}{d\Sigma}\right)^{1/2}$ is the gas sound speed. To derive this dispersion relation you may assume that

$$\Phi' = -\frac{2\pi G\Sigma'}{|k|} \,, \tag{2}$$

and consider only perturbations with short radial wavelength such that $kR \gg 1$.

Assuming now a Keplerian potential calculate the epicyclic frequency κ, and write down a condition for ω for the gravitational instability to develop. By analyzing the roots of the dispersion relation for k, show that the following condition must be satisfied:

$$Q = \frac{c_{\rm s}\Omega}{\pi G\Sigma} < 1 \,, \tag{3}$$

where Q is the Toomre parameter.

By re-writing equation (1) in the following form:

$$\frac{\omega^2}{\Omega^2} = 1 - \frac{2h|k|}{Q} + h^2k^2 \,, \tag{4}$$

where $h = c_{\rm s}/\Omega$, interpret the physical meaning of each term on the right-hand side of this equation to show what may stabilize the gravitational instability on small and large spatial scales, respectively.

[Hint: Recall that the relevant R- and ϕ-components of the convective derivative in the momentum equation are: $u_R\frac{\partial u_R}{\partial R} - \frac{u_\phi^2}{R}$ and $u_R\frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}$, respectively.]

2 (a) Consider a steady, geometrically thin and optically thick accretion disc around a black hole with mass M. The radial and azimuthal components of the Navier-Stokes equation read:

$$u_R \frac{\partial u_R}{\partial R} - \frac{u_\phi^2}{R} + \frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{GM}{R^2} = 0\,, \tag{1}$$

$$\Sigma\left(u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{d\Omega}{dR}\right), \tag{2}$$

where R is the cylindrical radius, u_R is the radial velocity, u_ϕ is the azimuthal velocity, ρ is the density of the disc, p is the pressure of the disc, G is the gravitational constant, Σ is the surface density of the disc, ν is the kinematic viscosity and Ω is the angular velocity of the disc. Using the azimuthal component of the Navier-Stokes equation together with the mass conservation equation derive how Σ and u_R depend on the mass accretion rate, $\dot{m}$, ν and R.

(b) Explain in detail what is meant by the Shakura and Sunyaev α prescription and using physical arguments explain the relation of the α parameter to ν, disc scale-height H, and sound speed $c_{\rm s}$.

(c) Using leading order analysis of the thin disc equations demonstrate that u_R is highly sub-sonic and u_ϕ is highly super-sonic and physically interpret this finding.

(d) Assume now that the gas pressure is the dominant source of pressure in the disc, the radiative efficiency is 0.1, and that the kinematic viscosity ν scales as follows:

$$\nu \propto R^{3/4} f_{\rm Edd}^{3/10} \left(\frac{\alpha}{0.1}\right)^{4/5} \left(\frac{M}{10^6\,{\rm M}_\odot}\right)^{1/20}, \tag{3}$$

where $f_{\rm Edd}$ is the Eddington ratio and ${\rm M}_\odot$ is the Solar mass. Using the results from (a), or otherwise, calculate the disc mass, $M_{\rm d}(R)$, within R, in the limit where $R \gg R_{\rm ISCO}$, where $R_{\rm ISCO}$ is the radius of the innermost stable circular orbit to show that

$$M_{\rm d}(R) = C_1\, f_{\rm Edd}^{k_1} \left(\frac{\alpha}{0.1}\right)^{k_2} \left(\frac{M}{10^6\,{\rm M}_\odot}\right)^{k_3} \left(\frac{R}{R_{\rm S}}\right)^{k_4}, \tag{4}$$

where k_1, k_2, k_3 and k_4 are exponents to be determined and $R_{\rm S} = 2GM/c^2$ is the Schwarzschild radius, with c being the speed of light. You may assume that $C_1 \approx 10^{-2}\,{\rm M}_\odot$ without a need to calculate it.

(e) Recalling that the Toomre parameter, $Q(R)$, for this accretion disc may be written as:

$$Q(R) = \frac{\Omega c_{\rm s}}{\pi G \Sigma}\,, \tag{5}$$

determine how $Q(R)$ scales with R and show that there is a unique radius, $R_{\rm sg}$, the self-gravitating radius, where $Q(R_{\rm sg}) = 1$.

Hence, derive:

$$\frac{R_{\rm sg}}{R_{\rm S}} = C_2\, f_{\rm Edd}^{k_5} \left(\frac{\alpha}{0.1}\right)^{k_6} \left(\frac{M}{10^6\,{\rm M}_\odot}\right)^{k_7}, \tag{6}$$

where k_5, k_6, k_7 are exponents to be determined. You may assume that $C_2 \approx 10^5$ without a need to calculate it.

[QUESTION CONTINUES ON THE NEXT PAGE]

By comparing $R_{\rm sg}$ to $R_{\rm ISCO}$, and assuming $\alpha \sim 0.1$ and $f_{\rm Edd} \sim 1$, show that there is a critical black hole mass for which a thin disc cannot exist. What is the physical meaning of this result? By comparing the likely Eddington luminosity of this critical black hole mass with the observed quasar luminosities what do you deduce? Can there be black holes more massive than this critical mass in the Universe?

3 (a) Explain what radiatively inefficient black hole accretion means and comment how this may occur both at high and low Eddington ratios.

Furthermore, by considering that the net energy advection rate, $q_{\rm adv}$, is given by the balance of the volume heating and cooling rates, q_+ and q_-, respectively, explain which three types of solutions are possible and to which types of accretion flows they correspond.

(b) Consider the radial component of the momentum equation in the following form:

$$(\vec{v}_{\rm p} \cdot \nabla)\vec{v}_{\rm p} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \Omega^2\vec{R}\,, \tag{1}$$

where $\vec{v}_{\rm p}$ is the poloidal velocity vector, ρ is the gas density, p is the gas pressure, Φ is the gravitational potential, Ω is the angular velocity and $\vec{R}$ is the cylindrical radius vector. Write down the surviving terms of this equation for steady flows that correspond to: i) thin Keplerian discs, ii) stellar atmospheres, iii) gravitational collapse, iv) slim discs, v) thick discs, vi) Bondi-Hoyle accretion and vii) sub-Keplerian ADAFs.

(c) Recall that the viscous dissipation for a steady, standard thin accretion disc can be written as:

$$F_{\rm diss}(R) = \frac{3GM\dot{m}}{4\pi R^3}\left[1 - \left(\frac{R_{\rm ISCO}}{R}\right)^{1/2}\right], \tag{2}$$

where G is the gravitational constant, M is the mass of the black hole, $\dot{m}$ is the mass flux through the disc, R is the cylindrical radius and $R_{\rm ISCO}$ is the radius of the innermost stable circular orbit. Hence, calculate the total luminosity of the disc between $R_{\rm ISCO}$ and $R_2 \to \infty$ and compare it with the accretion luminosity, $L_{\rm acc}$. What do you deduce from this comparison?

(d) Consider now a slim accretion disc in a steady state. If the vertically-averaged azimuthal component of the Navier-Stokes equation can be written as:

$$\Sigma R u_{\rm R}\frac{d(R^2\Omega)}{dR} = \frac{d}{dR}\left(\nu\Sigma R^3\frac{d\Omega}{dR}\right), \tag{3}$$

where R is the cylindrical radius, u_R is the radial velocity, Σ is the surface density of the disc, ν is the kinematic viscosity and Ω is the angular velocity of the disc, show that:

$$\mathcal{G}_2 - \mathcal{G}_1 = -\dot{m}(l_2 - l_1)\,. \tag{4}$$

Here, $\dot{m}$ is the mass flux through the disc, and $\mathcal{G}(R)$ and $l(R)$ are the torque and the specific angular momentum, evaluated at two radii in the disc, R_1 and R_2.

As viscous dissipation can be expressed as:

$$F_{\rm diss} = \frac{1}{2\pi R}\mathcal{G}(R)\frac{d\Omega}{dR}\,, \tag{5}$$

derive that the total rate of energy generation by viscosity, $L_{\rm gen}$, between $R_{\rm in} \sim R_{\rm ISCO}$ and $R_{\rm out}$ can be written as:

$$L_{\rm gen}(R_{\rm in}, R_{\rm out}) \simeq \dot{m}[e_{\rm out} - e_{\rm in} - \Omega_{\rm out}(l_{\rm out} - l_{\rm in})]\,. \tag{6}$$

Here, e is the gas specific energy and at the surface of the accretion disc you may assume that the difference in the rotational potentials, $\psi_{\rm rot,2} - \psi_{\rm rot,1}$, between radii R_2 and R_1, is equal to the difference in the gravitational potentials, $\Phi_2 - \Phi_1$, with $\nabla\psi_{\rm rot} = \Omega^2(R)\vec{R}$.

[QUESTION CONTINUES ON THE NEXT PAGE]

In the limit $R_{\rm in} \ll R_{\rm out}$ find a simple expression to relate $L_{\rm gen}$ to the total luminosity of the standard thin accretion disc as calculated in (c), which is a function of the angular velocity at $R_{\rm in}$, and comment on the physical meaning of this expression.

END OF PAPER