UNIVERSITY OF CAMBRIDGE

MAMA/345, NST3AS/345, MAAS/345

MAT3

# MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 9:00 am to 12:00 pm

## PAPER 345

## FLUID DYNAMICS OF THE ENVIRONMENT

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Consider a bi-disperse particle suspension of spherical particles of densities $\rho_1$, $\rho_2$ and volume concentrations $\phi_1$, $\phi_2$, respectively. The corresponding Stokes settling velocities of individual particles are $\widehat{W}_1$, $\widehat{W}_2 < 0$ and $z$ is vertically upwards.

(a) Suppose the suspension is horizontally homogeneous such that $\phi_i = \phi_i(z,t)$ (for $i = 1, 2$), the fluid is quiescent other than the motion due to the settling of the particles, and changes to the bulk viscosity due to the presence of the particles can be neglected. Show that for a 'hindered settling' regime, the particle settling velocities are given by

$$W_1 = (1-\phi_1)\widehat{W}_1 - \phi_2\widehat{W}_2\,,$$
$$W_2 = (1-\phi_2)\widehat{W}_2 - \phi_1\widehat{W}_1\,,$$

if the upward motion of the fluid displaced by the settling particles is felt equally by all the particles.

(b) Consider the particles settling in a horizontally homogeneous quiescent fluid.

(i) Give expressions for the one-dimensional time-dependent conservation relations for the volume concentrations $\phi_1$, $\phi_2$. Determine the characteristics $\lambda_j$ ($j = 1, 2...$) of this system of equations.

(ii) Consider the case $\widehat{W}_1 = (1-\epsilon)\widehat{W}$ and $\widehat{W}_2 = (1+\epsilon)\widehat{W}$ for $\epsilon \ll 1$. Determine the characteristics $\lambda_j$ to $\mathcal{O}(\epsilon)$. Explain the meaning of the leading-order term for each characteristic and give the corresponding leading-order ordinary differential equation for the evolution of the particle concentration along it.

(c) Consider a deep ambient fluid in a horizontal channel of unit width and rectangular cross section. A high-Reynolds-number Boussinesq shallow water flow is created by the instantaneous release of a bi-disperse suspension at one end of the channel in a region of length $L_0$ and depth $h_0$. The suspension has uniform initial particle volume concentrations $\phi_1(t=0) = \widetilde{\phi}_1$ and $\phi_2(t=0) = \widetilde{\phi}_2$, with $\widetilde{\phi}_1$, $\widetilde{\phi}_2 \ll 1$, and the fluid containing the particles is the same as that elsewhere in the channel. You may assume the interior of the current remains well mixed and of constant volume, and that any particles settling onto the base of the channel remain on the base.

(i) State a suitable condition for the front of the current that develops and derive an integral (box) model for the long-time evolution of the length $L(t)$ and depth $h(t)$ of the current.

(ii) In the case $\widehat{W}_1 = \widehat{W}_2 = \widehat{W}$, show that

$$L_\infty = \left[L_0^{5/2} + C\frac{(L_0 h_0)^{3/2}\widetilde{g}'^{1/2}}{|\widehat{W}|}\right]^{2/5},$$

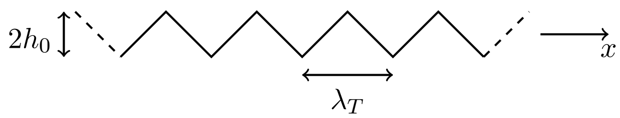
giving a suitable form for the initial total reduced gravity $\widetilde{g}'$ and determining the constant $C$.

(iii) How will the run-out length change if $\widehat{W}_1 = (1-\epsilon)\widehat{W}$ and $\widehat{W}_2 = (1+\epsilon)\widehat{W}$ for $0 < \epsilon \ll 1$ when $\widetilde{\phi}_1\rho_1 < \widetilde{\phi}_2\rho_2$? [*You should justify your answer but a detailed calculation is not required.*]

**2** Consider an incompressible Boussinesq stably stratified fluid of density $\hat{\rho}(z)$ (with $z$ directed vertically upwards) in which there is an internal gravity wave field of frequency $\omega$ and wavevector $\boldsymbol{k} = (k, 0, m)$.

(a) Starting from the Navier–Stokes equations and mass conservation, derive the equations governing the internal gravity waves and hence determine the dispersion relationship for a plane wave with suitably small kinematic viscosity $\nu$ and mass diffusivity $\kappa$. Show that the plane-wave solution satisfies the nonlinear equations. Describe briefly how nonzero $\nu$ or $\kappa$ affects the structure of the wave field (you need not give a full derivation but you should explain the structure).

(b) Consider the wave field reflected from a boundary centred on $z = 0$. The boundary has a symmetric sawtooth profile (see figure) characterised by amplitude $h_0$ and wavelength $\lambda_T$. The incoming wave field is linear with vertical velocity $w_i(\boldsymbol{x}, t) = \tilde{w}_i \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$.




(i) For what range of frequency $\omega$ will the reflection be subcritical? Using ray tracing, sketch this situation.

(ii) Sketch the behaviour if the reflection is not subcritical. Is this behaviour likely to persist? [*You should justify your answer.*]

(c) Suppose the boundary used for part (b) is replaced by a sinusoidal one, also centred on $z = 0$ but now with height profile $h = h_0 \sin k_T x$, where $k_T = 2\pi/\lambda_T$.

(i) For what range of $\omega$ will the reflection be subcritical? For a subcritical reflection in the limit $\nu = \kappa = 0$, determine to $\mathcal{O}(h_0^2)$ the vertical velocity $w_r(\boldsymbol{x}, t)$ of the reflected wave field. State any requirements for the result to be valid.

(ii) Sketch the behaviour if the reflection is not subcritical. [*You need not compute the reflected wave field.*]

**3** Consider a well insulated room of floor area $A$ and height $H$. The room has a single vent of height $H'$ spanning the length $L$ of one wall of the room. The floor is located at $z = 0$ and the lower edge of the vent is located at $z = z_V$. A point source of buoyancy flux $B > 0$ is located on the floor in the centre of the room. Assume that a two-layer stratification develops and reaches a steady state with an upper-layer density $\rho_H$ and lower-layer density $\rho_L$ separated by an interface at height $z = h$ above the floor. Assume also that $h < z_V$, there is no wind, the external density $\rho_1$ (away from the vent) is constant, $\rho_1 > \rho_L > \rho_H$, and the flow is incompressible.

(a) Explain the Boussinesq approximation and Batchelor's entrainment hypothesis. Discuss their use in the context of this problem.

(b) Determine the exchange flow rate $Q_V$ through the vent in terms of the densities involved, stating any assumptions made.

(c) Assume the point source at the floor generates a pure Boussinesq axisymmetric plume with top-hat profiles. State the necessary governing equations and determine the plume's radius $r$, top-hat velocity $U$, volume flux $Q_B$, and the reduced gravity $g'_B$ as functions of $z$.

(d) Assume the cold air entering through the vent descends along the wall beneath the vent as a pure Boussinesq line plume with top-hat profiles and negligible horizontal momentum. Determine the height of the virtual origin $z = z_o$ for this plume if the plume width is $b = H'/2$ at $z = z_V$. Determine (as functions of $z$) the velocity $W$ within the plume, the width $b$ of the plume, and the reduced gravity $g'$.

(e) Establish the implicit relationship between the geometry of the room, the interface height $h$ and the buoyancy flux $B$ from the point source. Determine the corresponding exchange flow rate $Q_V$. [*Hint: Consider a heat balance.*]

(f) Find an implicit relationship for the steady interface height when a second vent of same size and vertical position on the oppoosite wall is opened.

## END OF PAPER