MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 1:30 pm to 3:30 pm

PAPER 342

TOPOLOGICAL QUANTUM MATTER

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 This question discusses the two-dimensional surface code, the charge-flux model of anyons, and Chern-Simons theory.

(a) Define e and m particles in the surface code and show that they obey the fusion rules $e \times e = \mathbf{1}$, $m \times m = \mathbf{1}$, where $\mathbf{1}$ is the vacuum, or "identity", particle. Show that encircling an e with an m or vice versa yields phase $\exp(2i\theta_{em}) = \exp(2i\theta_{me}) = -1$. Show that exchanging two e particles yields $\exp(i\theta_{ee}) = 1$ and that the same holds for m particles, $\exp(i\theta_{mm}) = 1$.

(b) Upon fusion, e and m yield the f particle: $e \times m = f$. Show that exchanging two f particles yields $\exp(i\theta_{ff}) = -1$. Compute the spin of the f particle and comment on whether it is consistent with the spin-statistics relation.

(c) Define the charge-flux model of anyons and show that encircling a particle (q_1, Φ_1) by a particle (q_2, Φ_2), where q_j and Φ_j are particle charges and fluxes, respectively, yields $\exp(2i\theta_{12})$ with $2\theta_{12} = (q_1\Phi_2 + q_2\Phi_1)/\hbar$. (You may invoke the Aharonov-Bohm and Aharonov-Casher effects without proof.)

Suppose that the charge-flux model has charge quantised in units of q_0 and flux quantised in units of Φ_0. Furthermore, suppose that particles can be detected only via encircling them by other particles; hence we identify particle n with $\mathbf{1}$ if $\exp(2i\theta_{np}) = 1$ for all particles p.

Suppose that $a = (q_0, 0)$ and $b = (0, \Phi_0)$ satisfy $\exp(2i\theta_{ab}) = -1$. Show that $a \times a = \mathbf{1}$, $b \times b = \mathbf{1}$, and that $c = a \times b$ satisfies $\exp(i\theta_{cc}) = -1$. Hence interpret e and m of the surface code in terms of the charge-flux model.

(d) Certain topologically ordered systems can be characterised by a 2×2 matrix K and the assignment of 2-component vectors $\mathbf{q}$ to particles. Both K and $\mathbf{q}$ have integer entries. Fusing particles $\mathbf{q}$ and $\mathbf{q}'$ yields $\mathbf{q} + \mathbf{q}'$. Encircling particle $\mathbf{q}$ by particle $\mathbf{q}'$ yields $\exp(2i\theta_{qq'})$ with $\theta_{qq'} = \pi \mathbf{q}' \cdot K^{-1}\mathbf{q}$.

By computing braiding and fusion properties, show that the surface code can be described by $K = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$, with $\mathbf{q}_e = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{q}_m = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for e and m particles, respectively. In inspecting fusion, similarly to part (c), you may assume that particles can be detected only through encircling them by other particles.

2 This question is on the stabilizer formalism and on quantum error correction with the surface code.

(a) A stabilizer group $\mathcal{S}$ is a subgroup of the n-qubit Pauli group $\mathcal{P}_n$ such that $-\mathbb{1} \notin \mathcal{S}$; it defines a codespace $\mathcal{L} \subset \mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ as $\mathcal{L} = \{|\psi\rangle \in \mathcal{H} \mid g|\psi\rangle = |\psi\rangle, \forall g \in \mathcal{S}\}$.

Show that if $\mathcal{S} = \langle S_1, S_2, \ldots, S_m \rangle$, with generators $S_1, S_2, \ldots, S_m$, then $S_j^2 = \mathbb{1}$ and $S_j = S_j^\dagger$, $\forall j$. Further show that $\dim \mathcal{L} > 0$ implies that $\mathcal{S}$ is Abelian. Define the centralizer $\mathcal{C}(\mathcal{S})$.

(b) Consider a stabilizer code based on $\mathcal{S}$. Describe error detection and define the syndrome $\mathbf{s}$. Suppose that the code suffered an unknown error from $\mathcal{P}_n$, yielding $\mathbf{s}$. Explain why the only information that $\mathbf{s}$ reveals about the error is that it belongs to $E_\mathbf{s}\, \mathcal{C}(\mathcal{S})$, where $E_\mathbf{s} \in \mathcal{P}_n$ is an error consistent with $\mathbf{s}$.

The rest of the question is about the surface code on the square lattice, with qubits on the links. Suppose that the code furnishes a single logical qubit with logical Pauli operators $\bar{Z} = \prod_{j \in \gamma} Z_j$ and $\bar{X} = \prod_{j \in \tilde{\gamma}} X_j$ for suitable noncontractible paths γ and $\tilde{\gamma}$. Suppose furthermore that the code is subject to the bit-flip channel, i.e., it suffers errors X_j with probability p (with $0 \leqslant p < 1/2$) independently on each qubit j.

(c) Relate the probability that an error belongs to $E_\mathbf{s} \bar{X}^q \mathcal{S}$, with $q \in \{0, 1\}$, to the partition function $\mathcal{Z}_{\mathbf{s},q}$ of the two-dimensional Ising model

$$H_{\mathbf{s},q} = -J \sum_{vv',\, \text{n.n.}} \eta_{vv'}(\mathbf{s}, q) \sigma_v \sigma_{v'},$$

where $e^{\beta J} = \sqrt{(1-p)/p}$ with β the inverse temperature, $\sigma_v \in \{1, -1\}$, and the sum is over nearest-neighbour (n.n.) v, v'. Define $\eta_{vv'}(\mathbf{s}, q)$ in terms of $E_\mathbf{s} \bar{X}^q$.

(d) The Ising Hamiltonian $H_{\mathbf{s},q}$ may need to be modified at boundaries. Consider the surface code on a finite cylinder. Suppose that γ in $\bar{Z}$ connects the cylinder's two boundaries. Define boundary stabilizer generators that allow for such $\bar{Z}$ and show that the system indeed furnishes a single logical qubit. Derive the modifications of $H_{\mathbf{s},q}$ accounting for the boundaries.

3 The mean field theory of superconductors involves the Bogoliubov-de Gennes (BdG) Hamiltonian H_{BdG} which, in position representation, has structure

$$H_{\mathrm{BdG}} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^* \end{pmatrix}$$

with matrices h and Δ. In position representation, particle-hole (PH) symmetry is $\Sigma_1 H^*_{\mathrm{BdG}} \Sigma_1 = -H_{\mathrm{BdG}}$, where Σ_1 is the first Pauli matrix in the 2×2 grading of H_{BdG}.

(a) Show that any quadratic fermion Hamiltonian $H = \sum_{ij} h_{ij} a_i^\dagger a_j + \frac{1}{2}\Delta_{ij} a_i^\dagger a_j^\dagger + \frac{1}{2}\Delta^*_{ij} a_j a_i$ can be written as $H = i \sum_{jk} A_{jk} c_j c_k$, up to a constant, with a real antisymmetric matrix A and Majorana fermions c_j.

(b) Show that, for a translation-invariant system, the momentum-space BdG Hamiltonian satisfies $H_{\mathrm{BdG}}(\mathbf{k}) = -\Sigma_1 H^*_{\mathrm{BdG}}(-\mathbf{k}) \Sigma_1$ where $\mathbf{k}$ is the wave vector. Derive the relation this imposes between the BdG energy spectra $\{\varepsilon_{\mathbf{k}}\}$ and $\{\varepsilon_{-\mathbf{k}}\}$.

(c) A two-dimensional (2D) topological superconductor furnishes a boundary mode such that $\varepsilon_{k=0} = 0$ and $\partial_k \varepsilon_k > 0$ for all k, where k is the wave vector along the boundary. Explain why this is a one-way mode. Show that the corresponding long-wavelength Hamiltonian is

$$H^{(\mathrm{lw})} = -i\hbar v \int \mathrm{d}x\, c\, \partial_x c, \quad \hbar v = (\partial_k \varepsilon_k)_{k=0},$$

where x is the coordinate along the boundary. Show that $c(x)$ is a Majorana field. In $H^{(\mathrm{lw})}$, what plays the role of A from part (a)?

(d) A vortex in this 2D superconductor can be modeled as follows: We cut the superconductor into two halves along a line, thus creating two adjacent boundaries, the first with Majorana field $c_1(x)$ and the second with $c_2(x)$, with x the coordinate along the cut. Then, we glue the two halves back together by coupling the two boundaries via $2it \cos[\phi(x)/2]\, c_1(x) c_2(x)$, where the phase difference $\phi(x)$ between the two halves increases from $\phi(x < -R_{\mathrm{c}}/2) = 0$ to $\phi(x > R_{\mathrm{c}}/2) = 2\pi$ across a "vortex core" of size R_{c} around $x = 0$. The coupled boundaries have Hamiltonian $H^{(\mathrm{lw})} = \int \mathrm{d}x\, h$ with

$$h = i\hbar v_1\, c_1 \partial_x c_1 + i\hbar v_2\, c_2 \partial_x c_2 + 2it \cos(\phi/2)\, c_1 c_2.$$

Explain why $\mathrm{sgn}(v_1) = -\mathrm{sgn}(v_2)$. Assuming $|v_1| = |v_2| = v$, show that the vortex binds a Majorana zero mode (MZM). Estimate the spatial extent of the MZM in terms of the system parameters v, t, and R_{c}. Argue that the MZM remains present even if $|v_1| \neq |v_2|$.

(e) A finite superconductor always has an even number of MZMs. Consider a finite disk of the 2D superconductor in part (c) and suppose that the system has M vortices. Deduce how the boundary conditions for $c(x)$ must depend on M.

END OF PAPER