UNIVERSITY OF CAMBRIDGE

MAMA/341, NST3AS/341, MAAS/341

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 1:30 pm to 4:30 pm

PAPER 341

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **THREE** questions from Section A and **ONE** question from Section B. Each question from Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Given the ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^d$, we consider the two-step method

$$\mathbf{y}_{n+2} - h\mathbf{f}(\mathbf{y}_{n+2}) + \frac{2}{5}h^2 \frac{\partial \mathbf{f}(\mathbf{y}_{n+2})}{\partial \mathbf{y}} \mathbf{f}(\mathbf{y}_{n+2}) = \frac{4}{5}\mathbf{y}_{n+1} + \frac{1}{5}\mathbf{y}_n + \frac{1}{5}h\mathbf{f}(\mathbf{y}_n).$$

a. What is the order of the method?

b. Is the method convergent? *[Hint: You may assume that the Dahlquist equivalence theorem remains true in this setting.]*

c. Is the method A-stable?

2

Let $c_1, c_2 \in [0,1]$ be distinct numbers and $\omega(x) = (x - c_1)(x - c_2)$.

a. Construct the two-stage Runge–Kutta method corresponding to collocation at the points c_1 and c_2.

b. Determine the conditions on ω for the method to be of order at least 3.

c. Let $c_1 = \frac{1}{2}$. Form the matrix M and determine the values of c_2 (if any) for which the method is algebraically stable.

3

The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t \geqslant 0,$$

given with an initial condition for $t = 0$ and Cauchy boundary conditions, is solved by the fully discretised method

$$-\mu u_{m-1}^{n+1} + (1 + 2\mu)u_m^{n+1} - \mu u_{m+1}^{n+1} = u_m^n,$$

where μ is the Courant number.

a. What is the order of the method?

b. What is the range of Courant numbers for which it is stable?

c. Suppose that the equation is given for $x \in [0,1]$ with zero boundary conditions. What is the range of Courant numbers for which it is stable?

4

a. Let $p, q, r \in L_2[-1,1]$, where $p(x) > 0$, $q(x), r(x) \geqslant 0$ for $x \in [-1,1]$ and set

$$\mathcal{L}[u] = [p(x)u''(x)]'' - [q(x)u'(x)]' + r(x)u(x).$$

Prove that $\mathcal{L}$, given with zero boundary conditions at the endpoints, is a positive-definite operator acting on a Sobolev space $\mathcal{H}$ which you should specify explicitly.

b. Determine a variational problem which is minimised by the weak solution of the equation $\mathcal{L}[u] = f \in C[-1,1]$ with zero boundary conditions at $x = \pm 1$.

c. Propose a suitable set of finite-element functions $\Phi = \{\phi_1, \ldots, \phi_n\} \subset \mathcal{H}$ and determine a linear system of equations for the vector $[a_1, \ldots, a_n]$ so that

$$u_n(x) = \sum_{k=1}^{n} a_k \phi_k(x)$$

is a Ritz approximation to the solution of $\mathcal{L}u = f$ using Φ.

5

We consider the *linear Schrödinger equation*

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + V(x)u, \qquad x \in [-1,1], \quad t \geqslant 0,$$

where V is a given real function, given a complex-valued initial value at $t = 0$ and periodic boundary conditions at $x = \pm 1$.

a. Prove that the Euclidean norm of the solution u stays constant as t varies,

$$\int_{-1}^{1} |u(x,t)|^2 dx = \text{const.}$$

b. We solve the equation using the Strang splitting, composing the solutions of

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \qquad \text{and} \qquad i\frac{\partial u}{\partial t} = V(x)u.$$

In the first equation we semidiscretize using finite differences on an equidistant grid, approximating

$$\frac{\partial^2 v(m\Delta x)}{\partial x^2} \approx \frac{v((m+1)\Delta x) - 2v(m\Delta x) + v((m-1)\Delta x)}{(\Delta x)^2}$$

and solving exactly a linear system with a TST matrix, while we use the exact solution of the second equation. What is the order of the method?

c. Prove that the Euclidean norm of the numerical solution $\mathbf{u}^n$ stays constant as n varies,

$$\sum_m |u_m^n|^2 = \text{const.}$$

SECTION B

6

Write an essay on stability analysis of time-dependent finite-difference methods using eigenvalue analysis. Explain the uses and limitations of this approach and provide main proofs. Present two examples: (i) an implicit fully discretized method whose stability can be determined by eigenvalue analysis, and (ii) a fully discretized method for which eigenvalue analysis gives a wrong result.

7

Write an essay on splitting methods for linear time-dependent PDEs. You should explain (with examples) the rationale and advantages of this approach, provide examples of popular splittings, determine their order and explain how to increase the order of a splitting for time-symmetric methods.

END OF PAPER