UNIVERSITY OF CAMBRIDGE

MAMA/337, NST3AS/337, MAAS/337

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 11 June 2024 9:00 am to 11:00 am

PAPER 337

APPLICATIONS OF QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a system with d spatial dimensions with a conserved particle number N. The possible spontaneous breaking of particle number symmetry can be captured by the effective Landau-Ginzburg Lagrangian density for the complex order parameter Ψ:

$$\mathcal{L} = i\Psi^\dagger \partial_t \Psi - \frac{1}{2m}\partial_i \Psi^\dagger \partial_i \Psi - r\Psi^\dagger\Psi - \lambda(\Psi^\dagger\Psi)^2 + \cdots .$$

(a) Write down the generator N in terms of Ψ. Set $\Psi = \sqrt{\rho}e^{-i\theta}$ and obtain the effective Lagrangian density for ρ and θ. What is the physical meaning of ρ? What is the momentum conjugate to the phase θ? Obtain an uncertainty relation between the spatially uniform phase $\theta_0 \equiv \int d^d x\, \theta$ and the particle number N. Briefly discuss the physical consequences of this uncertainty relation for spontaneous symmetry breaking.

(b) At a classical level, what condition on the Landau-Ginzburg Lagrangian implies spontaneous symmetry breaking? Suppose that spontaneous symmetry breaking occurs and set $\rho = \overline{\rho} + \delta\rho$, with $\overline{\rho}$ the stable value (that you will need to determine). Expand the Landau-Ginzburg Lagrangian to quadratic order in $\delta\rho$ and θ.

(c) Write down the path integral for the effective quadratic theory and integrate out the gapped mode $\delta\rho$. You may freely use any results you know about Gaussian integrals and it is sufficient to work to leading exponential order. Take a long wavelength expansion of your result and thereby obtain the effective Lagrangian for the Goldstone boson. Write down the velocity of the Goldstone boson in terms of the parameters of the original Landau-Ginzburg theory.

(d) Now heat the system up to temperatures $T > 0$. Let the classical symmetry breaking value of the phase be $\langle\theta\rangle = 0$. You may assume that the variance of the phase is given by

$$\langle\theta^2\rangle = \frac{1}{\chi}\int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{\omega_n^2 + v^2 k^2} .$$

Here χ and v are constants, the ω_n are Matsubara frequencies and k is the $d-$dimensional spatial wavevector. Show, by considering a suitable contour integral of $\frac{\coth(\pi z)}{z^2 - a^2}$ or otherwise, that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi \coth(\pi a)}{a} .$$

Thereby obtain an expression for $\langle\theta^2\rangle$ as a single integral over the magnitude k of the wavevector. Consider the possible dimensions $d = 1, 2, 3$. For which of these dimensions is the integral you have written finite? For which of these dimensions does spontaneous symmetry breaking occur at $T > 0$? Briefly justify your answer.

(e) By subtracting off the temperature-independent short distance divergence, or otherwise, use your integral expression for $\langle\theta^2\rangle$ to estimate the temperature T at which thermal phase fluctuations restore the symmetry in $d = 3$.

2

This question is concerned with a ferromagnetic phase of a lattice system with an $SU(2)$ spin symmetry.

(a) Explain briefly why the massless modes of the system will be described by a three-component vector $n(x,t)$, with $n \cdot n = 1$. What goes wrong if you try to write down a kinetic term in the effective Lagrangian for n that is first order in time derivatives?

(b) Let $n^i = z^\dagger \sigma^i z$, where the σ^i are Pauli matrices and z is a complex two-component vector obeying $z^\dagger \cdot z = 1$. What redundancy exists in this parametrisation? Show that an $SU(2)$ rotation of z corresponds to an $SO(3)$ rotation of n. Thereby write down a rotationally invariant first order kinetic term in terms of z (the Wess-Zumino term). Show that the redundancy of the parametrisation corresponds to adding a total derivative term to the effective Lagrangian. [You are not required to fix the coefficient of the Wess-Zumino term in terms of the microscopic spin].

(c) Using the parametrization

$$z = \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} ,$$

obtain an explicit form for the Wess-Zumino term in the Lagrangian in terms of ϕ and θ. Write down a suitable spatial derivative term to complete the leading order effective Lagrangian.

(d) Write $n = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and consider fluctuations $\phi = \delta\phi$ and $\theta = \frac{\pi}{2} - \delta\theta$. Obtain the quadratic effective Lagrangian for these fluctuations and thereby obtain the dispersion relation $\omega = \omega(k)$ for linearized spin waves.

(e) Write down the retarded Green's function $G^R_{\delta\theta\delta\theta}(\omega, k)$ for the spin waves and the corresponding spectral weight $\rho(\omega, k)$.

(f) Suppose that, at a small nonzero temperature T, the spectral weight does not change appreciably. State the fluctuation-dissipation theorem and use it to write down the corresponding structure factor $S(\omega, k)$ in the classical limit $\omega \ll T$. Perform a Fourier transform to obtain the real-time correlation function $C(t, k)$.

(g) Suppose that, at a slightly higher temperature, the spin wave acquires a decay rate $\Gamma(k)$, in addition to its dispersion $\omega(k)$ obtained above. You may model $\Gamma(k)$ as an additional imaginary part of the dispersion relation, but be careful to ensure a causal Green's function. Incorporating this effect, obtain the static structure factor

$$S(k) = \lim_{\omega \to 0} S(\omega, k) \propto \frac{T\omega(k)\Gamma(k)}{(\omega(k)^2 + \Gamma(k)^2)^2} .$$

The constant of proportionality depends on the normalisation of the Lagrangian.

(h) The static correlation function $C(x)$ is the Fourier transform of $S(k)$. Suppose that $\Gamma = \gamma k$ is linear in the wavevector, with γ a constant. Write down the leading large $|x|$ behaviour of $C(x)$ in this case – you need not perform the Fourier transform explicitly. Is this behaviour consistent with spontaneous symmetry breaking at this higher temperature?

END OF PAPER