UNIVERSITY OF CAMBRIDGE

MAMA/336, NST3AS/336, MAAS/336

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 11 June 2024 9:00 am to 11:00 am

PAPER 336

PERTURBATION METHODS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the integral

$$I(\lambda) = \frac{1}{\lambda^{1/2}} \int_0^\infty e^{-s+2\lambda-\lambda^2/s} ds.$$

(a) Show that, for $\lambda \in \mathbb{R}$, $0 < \lambda \ll 1$,

$$I(\lambda) \sim \frac{1}{\lambda^{1/2}} + 2\lambda^{1/2} + \lambda^{3/2}(1 + 2\ln\lambda + 2\gamma_e).$$

You may use without proof the identity

$$\int_0^\infty \ln(t) e^{-t} dt = -\gamma_e,$$

where γ_e is the Euler-Mascheroni constant.

[Hint: You will find it useful to divide the integral into two parts, one over the range $s = 0$ to δ^2, and the other from $s = \delta^2$ to ∞, where

$$0 < \lambda \ll \delta \ll 1,$$

and ignore terms containing δ^4 or higher powers of δ.]

(b) For $\lambda \in \mathbb{R}$, $\lambda \to \infty$, use the method of steepest descents to show that

$$I(\lambda) \sim \sqrt{\pi}.$$

You may use the result $\int_{-\infty}^{\infty} e^{-as^2} ds = \sqrt{\pi/a}$ without proof.

(c) If $\lambda = e^{i\pi/4}|\lambda|$, $|\lambda| \to \infty$, determine the location of the saddles and show that the steepest descents path satisfies

$$\frac{r}{|\lambda|}\sin\psi + \frac{|\lambda|}{r}\cos\psi = \sqrt{2},$$

where s has the polar representation $s = re^{i\psi}$. Sketch this contour.

2 (a) A weakly perturbed oscillator satisfies the equation

$$\frac{d^2u}{dt^2} + u = \epsilon f\left(u, \frac{du}{dt}, \epsilon t\right), \quad 0 < \epsilon \ll 1.$$

By using the method of multiple scales show that the leading-order solution takes the form $u \sim R(T)\cos\left(t + \phi(T)\right)$, where

$$\frac{dR}{dT} = -\left\langle f \sin(t+\phi)\right\rangle, \qquad R\frac{d\phi}{dT} = -\left\langle f\cos(t+\phi)\right\rangle,$$

in which $T = \epsilon t$ and $\langle \ldots \rangle$ denotes the average over the fast time period.

(i) Find $R(T)$ and $\phi(T)$ explicitly in the case

$$f = -u\left(\frac{du}{dt}\right)^2$$

subject to the initial conditions $u = 1$ and $du/dt = 0$ at $t = 0$.

(ii) Find $R(T)$ and $\phi(T)$ explicitly in the case

$$f = -u^2\frac{du}{dt}$$

subject to the initial conditions $u = 1$ and $du/dt = 0$ at $t = 0$.

(b) Derive the leading order solution of the WKB equation

$$\frac{d^2y}{dx^2} + \left(1 + (\epsilon x)^3\right)^2 y = 0, \quad 0 < \epsilon \ll 1,$$

which is uniformly valid in $x \geqslant 0$, with boundary conditions $y = 0$, $dy/dx = 1$ on $x = 0$.

[Hint: You may start by posing the WKB ansatz

$$y \sim \left(A_0(X) + \epsilon A_1(X) + \ldots\right)\exp\left(\pm\frac{i}{\epsilon}\int k(X)dX\right),$$

where $X = \epsilon x$ for some suitable $k(X)$.]

3 (a) In dimensionless form, the equation for the vibrational modes of a stretched beam occupying the region $-1 \leqslant x \leqslant 1$ is

$$\epsilon^2 \frac{d^4 y}{dx^4} - \frac{d^2 y}{dx^2} - \omega^2 y = 0.$$

The beam has low stiffness, i.e. $0 < \epsilon \ll 1$, and its ends are clamped:

$$y(\pm 1) = \frac{dy}{dx}(\pm 1) = 0.$$

Prove that the leading order symmetric eigenfunction takes the form

$$y \sim y_0 = a \cos \omega_0 x,$$

where the fundamental frequency is

$$\omega \sim \omega_0 = \pi/2.$$

Show that there is an inner region of size $O(\epsilon)$ about each end and determine the leading order inner solution. Match this to the leading, and second-order outer solution

$$y_1 = b \cos \frac{\pi x}{2} - \frac{\omega_1^2 a}{\pi} x \sin \frac{\pi x}{2}$$

(which you should derive), to show that the correction to the fundamental frequency of vibration is

$$\omega^2 \sim \frac{\pi^2}{4} + \frac{\pi^2}{2}\epsilon + O(\epsilon^2).$$

(b) Consider the equation

$$\epsilon \frac{d^2 y}{dx^2} + (1+x)^3 \frac{dy}{dx} + y = 0,$$

with $y(0) = y(1) = 1$. Find the leading terms in the inner and outer expansions as $\epsilon \to 0$, and write down an additive composite expansion which is uniformly valid to $O(1)$.

END OF PAPER