UNIVERSITY OF CAMBRIDGE

MAMA/335, NST3AS/335, MAAS/335

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 1:30 pm to 3:30 pm

PAPER 335

DIRECT AND INVERSE SCATTERING OF WAVES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let ψ be a time-harmonic acoustic field propagating in a 2-dimensional medium with refractive index $n = n(x, z)$, so that the wave equation for ψ can be written as

$$[\nabla^2 + k_0^2 n^2]\psi = 0 \ , \tag{1}$$

where k_0 is the wavenumber of a wave propagating in free space.

(i) Consider the field ψ

$$\psi(x, z) = E(x, z)e^{ik_0 x} \tag{2}$$

propagating at a small angle with respect to the x-direction, where $E(x, z)$ is a slowly-varying function of x, and derive the parabolic wave equation for E by factorising the differential operator in (1) using the operators

$$A = \frac{\partial}{\partial x} \qquad \text{and} \qquad B = \sqrt{\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2}$$

Explain under what condition it is a good approximation and what physical effects are being neglected.

(ii) Using the operators D and S defined by:

$$D = \frac{i}{2k_0}\frac{\partial^2}{\partial z^2} \quad , \quad S = \frac{ik_0}{2}(n^2 - 1) \ ,$$

and assuming that the operators D and S nearly commute over a small interval Δx, write an approximate expression for the solution to the parabolic equation, $E(x_0 + \Delta x, z)$ at a plane $x_1 = x_0 + \Delta x$, given in terms of $E(x_0, z)$.

Give conditions under which the assumption that D and S nearly commute is valid, and comment on any other approximations you need to derive this approximate solution.

Hence define a suitable $\tilde{E}_0(x_0, z)$ such that $E(x, z)$ is the approximate solution of the initial value problem

$$\frac{\partial E}{\partial x} = DE \qquad \text{in the interval } [x_0, x_0 + \Delta x] \tag{3}$$
$$\text{with initial condition } \tilde{E}_0(x_0, z) \ .$$

(iii) Explain how the solution $E(x_m, z)$ at a plane $x_0 + m\Delta x$, for a positive integer m, can be found using this method, given $E(x_0, z)$. *[You need not write the solution explicitly.]*

2 A time-harmonic wave $\psi(\mathbf{r})e^{-i\omega t}$ with wavenumber k_0, travelling in three-dimensional free space, is incident upon an inhomogeneity with refractive index $n(\mathbf{r})$ which occupies a volume D.

(i) Derive the first term of both the Born and the Rytov approximations for the space-dependent part of the total field at point $\mathbf{r}$, and denote them respectively by $\psi_B(\mathbf{r})$ and $\psi_R(\mathbf{r})$.

Show that the first order term in a power series expansion of the Rytov approximation is equal to the first-order Born approximation.

(ii) In the case where the incident field is a monochromatic plane wave propagating with wave number k_0 in a direction $\mathbf{r}_0$, derive far-field approximations to $\psi_B(\mathbf{r})$ and $\psi_R(\mathbf{r})$.

(iii) For the same incident wave as in (ii), assume that the refractive index in D is $n(\mathbf{r}) = 1 + W(\mathbf{r})$, where $W(\mathbf{r})$ is a statistically stationary random function of position with Gaussian p.d.f. and mean zero, and with variance $< W^2(\mathbf{r}) > \ll 1$.

The mean intensity in the Rytov approximation is given by

$$I(\mathbf{r}) = < \psi_R^*(\mathbf{r})\psi_R(\mathbf{r}) > \tag{1}$$

Derive an expression for $I(\mathbf{r})$ in the far field in terms of the autocorrelation function of the 'scattering potential' $V = k_0^2[n^2(\mathbf{r}) - 1]$.

[You may wish to use $Re(f) = \frac{1}{2}(f + f^*)$, for a complex function f; and the Taylor expansion when calculating $< \exp(\phi) >$ for a random phase ϕ.]

3 Given the equation

$$Ax = y \ , \tag{1}$$

where $A : X \to Y$ is a given compact linear operator between two Hilbert spaces, and $x \in X, \ y \in Y$, consider the inverse problem of finding x, given data y.

(i) Define the *Moore-Penrose generalised inverse operator* $A^\dagger$, and explain how $x^\dagger = A^\dagger y$ approximates the solution x to the problem given by (1), when it exists.

(ii) Define the Landweber iteration scheme for finding the solution to the normal equation associated with (1), and write an equation for the $(n+1)th$ iterate.

Given $y \in \mathcal{D}(A^\dagger)$, state a sufficient condition for the convergence of this Landweber iteration, and state its limit. *(You do not need to provide a proof.)*

(iii) Show that Landweber iteration is equivalent to minimising the functional

$$J(x) = \frac{1}{2} \parallel Ax - y \parallel^2 \tag{2}$$

with a gradient descent method. Here $\| \cdot \|$ denotes the L^2 norm.

(Recall: gradient descent is a method for finding a (local) minimum of a multivariable differentiable function (or functional) $F(x)$ using successive iterates $x_{n+1} = x_n - \gamma F'(x)$, with $\gamma \in \mathbb{R}^+$.)

(iv) By using the choice $x_0 = 0$, write the $(n+1)th$ iterate in Landweber iteration non-recursively as a finite sum $\sum_{k=0}^{n}$ of an appropriate expression.

Hence, relate a convergent Landweber iteration to the power expansion of $(A^*A)^{-1}$, where A^* is the adjoint of A.

END OF PAPER