UNIVERSITY OF CAMBRIDGE

MAMA/333, NST3AS/333, MAAS/333

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 1:30 pm to 4:30 pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Inertial oscillations on a β-plane

(a) Consider a local Cartesian coordinate system where x points to the east, y points to the north, and z is the local vertical direction. The angular velocity vector associated with the Earth's rotation can then be written $\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$ in Cartesian coordinates where θ is the latitude. State the *traditional* and β-plane approximations. If U and W are characteristic horizontal and vertical velocity scales, respectively, and L and H are characteristic horizontal and vertical length scales, explain the necessary conditions for the traditional and β-plane approximations to hold.

(b) Consider the movement of a drifter floating on the surface of the ocean in the absence of wind. Let the position of the drifter at time $t = 0$ be $(x, y) = (0, 0)$ in local Cartesian coordinates, and let the horizontal velocity of the drifter at $t = 0$ be $(u, v) = (0, V)$. Assume that the drifter moves with the local fluid velocity and assume that the fluid velocity is horizontally uniform. Invoking the traditional and β-plane approximations and clearly stating any other assumptions, show that the drifter position $(x(t), y(t))$ satisfies

$$\dot{x} = f_0 y + \frac{1}{2}\beta y^2, \tag{1}$$

$$\dot{x}^2 + \dot{y}^2 = V^2, \tag{2}$$

and provide physical interpretations for these two equations, where $(\dot{\ })$ denotes a time derivative. Hence, show that the north/south position of the drifter satisfies the following ordinary differential equation:

$$\dot{y}^2 = V^2 - \left(f_0 y + \frac{1}{2}\beta y^2\right)^2. \tag{3}$$

If the equator corresponds to the location where $f_0 + \beta y = 0$, find a necessary condition for the particle to stay in the Northern Hemisphere (with $f_0 + \beta y > 0$). *Note that you do not need to obtain explicit solutions to equation (3).*

(c) Let

$$\widetilde{\beta} \equiv \frac{\beta V}{f_0^2}. \tag{4}$$

For $\widetilde{\beta} \ll 1$, find explicit expressions for the position of the drifter described in part (ii) that is valid to $O(\widetilde{\beta})$. Show that this corresponds to circular motion superposed with a westward drift and write an expression for the speed of the westward drift. Provide a physical explanation for the westward drift, along with a sketch of the drifter trajectory, indicating the direction of travel.

2 Ekman pumping and Sverdrup balance

The incompressible Navier-Stokes equations for a fluid with a uniform density, ρ, in a rotating reference frame with Coriolis parameter, f, can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

(a) Consider a spatially variable wind stress applied to the surface of the ocean at $z = 0$ such that $\boldsymbol{\tau} = (\rho\nu u_z, \rho\nu v_z)$, where u and v are the x and y components of the velocity and the subscript denotes a partial derivative. Starting from the incompressible Navier-Stokes equations given above, derive an expression for the vertical velocity at a depth $z = -h$, which is sufficiently deep such that the viscous stress at $z = -h$ can be neglected. You may assume that the Rossby number is small. Clearly state any other assumptions.

(b) Now model the ocean interior as a shallow layer of fluid occupying the region between $z = -H$ and $z = -h$, where $H > h$ and with boundary conditions $w(z = -H) = 0$ and where $w(z = -h)$ is set from your solution in part (a). You may assume that the Rossby number is small and that the flow is inviscid in the ocean interior. Clearly state any other assumptions. If $f = f_0 + \beta y$, find the steady state solution for the velocity in the ocean interior.

(c) Derive an equation for the evolution of the shallow water potential vorticity in the ocean interior (the region between $z = -H$ and $z = -h$) with the same boundary conditions as given in part (b), where $H = H(x, y)$ and $h = h(x, y)$. Using this result or otherwise, sketch the steady circulation with small Rossby number in a situation where $\nabla(H - h) = (c, 0)$, c is a positive constant, and $w(z = -H) < 0$. Explain your reasoning and discuss the physical mechanisms associated with this flow.

3 Flow over a ridge – Atmospheric internal gravity waves

(a) Write down the Boussinesq equations for stably stratified, ideal-fluid flow in an inertial (non-rotating) frame of reference and briefly state the conditions for their validity. Linearize the equations about a basic state in which the buoyancy frequency is $N = N(z)$, where z is vertically upwards, and the flow is in the positive x direction with velocity $(\overline{u}(z), 0, 0)$, only considering two-dimensional disturbances that depend on x, z, t ($v = 0$ w.l.o.g.).

(b) Show that

$$D_t(u'_z - w'_x) + \overline{u}_{zz} w' + \sigma'_x = 0, \tag{1}$$

where σ' is the disturbance buoyancy, D_t is the linearized material derivative, $D_t = \partial_t + \overline{u}\,\partial_x$, and the other symbols are disturbance fields in standard notation. Deduce that

$$D_t^2\left[w'_{xx} + w'_{zz}\right] - \overline{u}_{zz} D_t w'_x + N^2 w'_{xx} = 0, \tag{2}$$

(c) Consider stationary wave disturbances of the form $w' = \hat{w}(z)\exp(ikx - i\omega t)$ (real part implied). Show that $\hat{w}$ satisfies

$$\hat{w}_{zz} + m^2(z)\hat{w} = 0, \tag{3}$$

where

$$m^2(z) = l^2(z) - k^2, \tag{4}$$

and with $l(z)$ given by

$$l^2(z) = \frac{N^2(z)}{\left[\overline{u}(z) - \frac{\omega}{k}\right]^2} - \frac{\overline{u}_{zz}}{\left[\overline{u} - \frac{\omega}{k}\right]}. \tag{5}$$

(d) Consider waves where the horizontal phase velocity of the wave is $c = \omega/k = 0$. Assuming that the vertical scale on which $l(z)$ varies is larger than that of the wave disturbances and imposing suitable boundary conditions, briefly discuss the qualitative nature of such disturbances including the tendency of the waves to be vertically trapped when $\overline{u}(z)$ increases with altitude, or when $N^2(z)$ decreases with altitude, or both.

(e) At the bottom boundary, $\hat{w}(0) = 0$. Show that

$$k^2 = \frac{\int_0^z (l^2\hat{w}^2 - \hat{w}_{z'}^2)\,\mathrm{d}z'}{\int_0^z \hat{w}^2\,\mathrm{d}z'}. \tag{6}$$

(f) Assume that k^2 is stationary with respect to small changes in $\hat{w}(z)$. By considering small changes dk and dc, derive a formula for the group velocity, $\partial\omega/\partial k$. Hence, deduce that when $l^2 > -N^2/\overline{u}^2$, waves generated by flow past a ridge will always appear downstream of the ridge.

4 Transformed Eulerian Mean equations

The Transformed Eulerian Mean equations are

$$\overline{u}_t - f_0\overline{v}_a^* = -\left(\overline{u'v'}\right)_y + \left(\frac{f_0\overline{\rho'v'}}{d\rho_s/dz}\right)_z, \tag{1}$$

$$f_0\overline{u} = -\frac{\overline{p}_y}{\rho_0}, \tag{2}$$

$$\overline{\rho}g = -\overline{p}_z, \tag{3}$$

$$\overline{v}_{a,y}^* + \overline{w}_{a,z}^* = 0, \tag{4}$$

$$\overline{\rho}_t + \overline{w}_a^*\frac{d\rho_s}{dz} = 0, \tag{5}$$

where symbols have their usual meanings and the subscripts x, y, z and t denote partial derivatives with respect to the spatial directions and time respectively. The subscript a denotes the ageostrophic component. ρ_s is the density variation in the hydrostatically balanced basic state.

(a) Briefly describe, without detailed calculations, how the Transformed Eulerian Mean equations are obtained from the Boussinesq β-plane primitive equations. Recall that the Eulerian mean velocities $(\overline{v}_a, \overline{w}_a)$ are related to the Transformed Eulerian Mean velocities by $(\overline{v}_a^*, \overline{w}_a^*)$ by

$$\overline{w}_a^* = \overline{w}_a + \frac{(\overline{\rho'v'})_y}{\mathrm{d}\rho_s/\mathrm{d}z}, \qquad \overline{v}_a^* = \overline{v}_a - \frac{\partial}{\partial z}\left[\frac{\overline{\rho'v'}}{\mathrm{d}\rho_s/\mathrm{d}z}\right].$$

(b) Define a streamfunction such that $(\overline{v}_a^*, \overline{w}_a^*) = (\overline{\chi}_{a,z}^*, -\overline{\chi}_{a,y}^*)$ and show that the Transformed Eulerian Mean equations can be written as

$$f_0^2\overline{\chi}_{a,zz}^* + N^2\overline{\chi}_{a,yy}^* = -f_0(\nabla\cdot\overline{\mathbf{F}})_z \tag{6}$$

where $\overline{\mathbf{F}} = (0, \overline{F}^{(y)}, \overline{F}^{(z)})$ is the zonally averaged Eliassen-Palm flux. You should give expressions for $\overline{F}^{(y)}$ and $\overline{F}^{(z)}$.

(c) For the remainder of this question, consider a flow at small Rossby number confined to a β-plane longitudinal channel with rigid walls at $y = 0, L$ and with $0 < z < \infty$. Waves are generated by topographic perturbations on the lower boundary. The buoyancy frequency, N, is assumed to be constant. The real part of the variation in the quasi-geostrophic streamfunction can be written in the form

$$\psi' = \Re\left[\hat{\psi}(z)e^{ikx}\sin\frac{\pi y}{L}\right].$$

Show that the Eliassen-Palm flux is purely vertical and of the form $\overline{F}^{(z)} = F_0\Theta(z)\sin^2\frac{\pi y}{L}$ where $F_0 = f_0^2/N^2$ and $\Theta(z)$ is a function of z.

[QUESTION CONTINUES ON THE NEXT PAGE]

(d) Now, assume instead that $\Theta(z)$ is a simple step function with

$$\Theta(z) = \begin{cases} 1 & z < H \\ 0 & z > H. \end{cases} \tag{7}$$

This is a simple representation of a situation where waves are generated far below $z = H$, propagate upwards, and dissipate in a thin critical layer localised about $z = H$. By expanding the wave forcing and the solution in a Fourier series or otherwise, solve Equation (6) for $\overline{\chi}_a^*$, stating suitable boundary conditions. Find expressions for the acceleration $\overline{u}_t$ and the rate of change of density $\overline{\rho}_t$.

(e) Derive an equation relating the Eulerian mean streamfunction to the Transformed Eulerian Mean streamfunction. Sketch the form of the response in $\overline{\chi}_a^*$, $\overline{u}_t$, $\overline{\rho}_t$ and $\overline{\chi}_a$ in the (y, z) plane. Comment on the response as seen in the Eulerian mean view versus the Transformed Eulerian Mean view, paying particular attention to the effects of vertical advection of the mean flow and eddy density fluxes.

(f) Suppose now that the region where the wave dissipates has a vertical scale D. Consider the dominant balances in the momentum equation when the shallow forcing limit where $\frac{ND}{f_0 L} << 1$ and the deep forcing limit $\frac{ND}{f_0 L} >> 1$ and comment on the nature of the response.

END OF PAPER