UNIVERSITY OF CAMBRIDGE

MAMA/332, NST3AS/332, MAAS/332

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 9:00 am to 12:00 pm

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Draining gravity current

A fluid with dynamic viscosity μ and density ρ flows as a two-dimensional gravity current over a sponge of thickness $h(p)$ and permeability $k(p)$ such that $k/h \propto p^{-\beta}$, where p is the overburden pressure on the top of the sponge and β is constant. The gravity current is supplied with fluid at the fixed horizontal location $x = 0$ with a volume flux proportional to t^{α}, where t is time and α is constant. The air above the gravity current and below the sponge are at equal pressures. The gravity current, whose thickness $H(x, t) \gg h$, therefore partially drains through the sponge as it flows. The sponge is supported by a fixed, rigid, porous base that allows fluid to pass through it freely.

(i) Calculate the vertical volume flux through the sponge as a function of H.

(ii) Starting from the equations for quasi-parallel flow and conservation of mass, derive a partial differential equation and boundary conditions sufficient to determine $H(x, t)$.

(ii) Find a functional relationship between α and β that allows the governing equations to admit a similarity solution, and write down the form of the similarity solution in terms of α.

(iv) For the particular case $\alpha = 2$, determine the ordinary differential equation and boundary conditions that describe the shape of the gravity current. Determine the functional form of the leading edge of the current.

2 Penetrative convection

Water has a nonlinear equation of state such that its density is given by

$$\rho = \rho_m \left[1 - \alpha(T - T_m)^2\right],$$

where ρ_m is the maximum density, achieved at temperature $T = T_m$ [4°C], and α is constant.

(i) Sketch the density field in a layer of water whose upper boundary has a temperature lower than T_m and whose lower boundary has a temperature greater than T_m. Give a brief explanation of where you would expect convection to occur.

A deep body of water has its upper surface maintained at temperature $T_s < T_m$. The water is stagnant and has a constant temperature gradient in $0 < z < h$, where z measures vertical distance below the surface. The water below $z = h$ has a uniform, mean temperature $T_w > T_m$ and convects turbulently, driven by the density difference between the water at $z = h$ and the water below.

(ii) By considering the local Rayleigh number associated with a boundary layer just below $z = h$, or otherwise, assuming that $h > 0$, determine an expression for the heat flux across $z = h$ in terms of T_h, T_m and T_w, where $T_h = T(z = h)$. [*You should define any variables that you introduce.*]

(iii) Maximise your expression for the heat flux with respect to T_h to show that the maximum convective heat flux is proportional to $(T_w - T_m)^{5/3}$. How does the constant of proportionality depend on the material properties of the system?

(iv) Assuming that the system adjusts itself to achieve this maximum flux, determine the steady thickness h of the stagnant layer as a function of T_w, given the boundary temperature T_s. Estimate the largest interior temperature of an ice-covered lake for which there is a stagnant layer of water below the ice.

(v) Now consider a lake of depth H with constant surface temperature $T_s < T_m$. Write the heat flux from the convecting region as $(k\Delta T/H)\mathcal{F}\theta(t)^{5/3}$, where k is the thermal conductivity of water, $\theta = (T_w(t) - T_m)/\Delta T$, $\Delta T = T_m - T_s$ and $\mathcal{F}$ is a constant, and consider the case that $\mathcal{F} \gg 1$. Write down a dimensionless equation for the time evolution of θ, in which the depth of the stagnant region is scaled with H and time is scaled with the diffusion time over a distance H. Given that $\theta(0) = 1$, determine that

$$\theta = \left(1 + \tfrac{2}{3}\mathcal{F}\tau\right)^{-3/2},$$

where τ is the scaled time.

(vi) For what values of θ and for what times does this solution remain valid as the leading order behaviour for $\mathcal{F} \gg 1$? Show that the Rayleigh number for the convecting region remains large while $1 - h/H$ is of order unity (not too small).

3 Icicle ripples

An icicle is modelled in two dimensions as a vertical sheet of ice at its freezing temperature T_m, having a thin film of melt water with unperturbed uniform thickness h flowing down it, surrounded by cold air with temperature less than T_m. The icicle grows laterally by freezing from the film of water. The rate of solidification is much smaller than the flow rate in the film so that the prescribed flux q of water in the film can be assumed to be constant and uniform. In Cartesian coordinates, the unperturbed surface of the ice is at $x = 0$, while z measures vertical distance downwards.

(i) Consider the flow in the film of water when the ice–film interface is perturbed to $x = \eta_1 e^{i\alpha z+\sigma t}$ and the film–air interface takes position $x = h + \eta_2 e^{i\alpha z+\sigma t}$, where η_1 and η_2 may be complex. Use the results of lubrication theory, linearised for $|\eta_{1,2}| \ll h$, to show that

$$h = \left(\frac{3\nu q}{g}\right)^{1/3}$$

and that

$$\eta_2 = \frac{\eta_1}{1 - i\Gamma\alpha^3 h},$$

where $\Gamma = \gamma/3\rho g$, γ is the water–air surface tension, ν and ρ are the kinematic viscosity and density of water, respectively, and g is the acceleration due to gravity.

(ii) Assuming that the unperturbed temperature field in the air is linear with gradient $(T_a - T_h)/\delta$ for some temperature T_a and length scale δ, where $T_h = T(x = h)$, show that

$$T_m - T_h = \frac{\epsilon}{1+\epsilon}(T_m - T_a), \quad \text{where} \quad \epsilon = \frac{k_a}{k_w}\frac{h}{\delta}$$

and k_a, k_w are the thermal conductivities of air and water respectively. Determine the unperturbed lateral solidification rate V of the icicle, taking care to identify all material parameters involved.

(iii) In what follows, take the limits $k_a/k_w \ll 1$, $\epsilon \ll 1$ with $\Gamma\alpha^3 h = \mathrm{O}(1)$, in addition to the thin-film approximation $\alpha h \ll 1$. [*Be careful to discard asymptotically small terms only in comparison with terms that are known to be* O*(1).*]

With the interfaces perturbed as above, consider perturbations to the temperature fields in the liquid film and the air, applying a quasi-stationary approximation and ignoring any advective heat transfer. You may assume that the ice–film interface remains at temperature T_m (*ignore the Gibbs-Thompson effect*) and that temperature and heat flux are continuous at the film–air interface.

Derive the complex dispersion relation

$$\frac{h}{V}\sigma = \frac{\alpha h}{1 - i\Gamma\alpha^3 h}.$$

Hence, determine the wavenumber α_m that has the largest growth rate in terms of Γ and h and the corresponding phase speed of the ensuing ripples in terms of V.

END OF PAPER