UNIVERSITY OF CAMBRIDGE

MAMA/331, NST3AS/331, MAAS/331

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2024 1:30 pm to 3:30 pm

PAPER 331

HYDRODYNAMIC STABILITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Euler's equations in cylindrical coordinates are (setting $\rho = 1$ for simplicity)

$$\frac{\partial u}{\partial t} + \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)u - \frac{v^2}{r} + \frac{\partial p}{\partial r} = 0,$$

$$\frac{\partial v}{\partial t} + \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)v + \frac{uv}{r} + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0,$$

$$\frac{\partial w}{\partial t} + \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)w + \frac{\partial p}{\partial z} = 0,$$

where the velocity components u, v and w are in the radial, azimuthal and axial directions respectively, p is the pressure and the condition for incompressibility is $\partial(ru)/\partial r + \partial v/\partial \theta + r\partial w/\partial z = 0$.

(i) Derive the linearised equations around the steady base state $(u, v, w) = (0, r\Omega(r), 0)$ (where $\Omega(r)$ is the angular velocity) for a small disturbance which has the normal mode form

$$\left[u'(r), v'(r), w'(r), p'(r)\right] \exp(\sigma t + im\theta + ikz). \qquad (*)$$

(ii) Reduce the system in (i) down to a second order equation for u' and hence derive Rayleigh's criterion for stability for the appropriate subset of disturbances (you can assume the flow is bounded between two solid walls at r_i and $r_o > r_i$).

(iii) A cylindrical vortex sheet located at $r = R_0$ has the associated flow

$$\Omega(r) = \begin{cases} 0 & 0 \leqslant r < R_0, \\ \Omega_0(R_0/r)^2 & R_0 < r < \infty. \end{cases}$$

Consider a 2D disturbance (with $\partial/\partial z = 0$ but general r and θ dependence) which is irrotational away from the vortex sheet by adopting the velocity potential $\phi(r, \theta, t)$ (where $\mathbf{u} = \nabla\phi$) so that

$$\phi = \begin{cases} \phi_1 & 0 \leqslant r < R_0, \\ \Gamma_0\theta + \phi_2 & R_0 < r < \infty. \end{cases}$$

where $\Gamma_0 = \Omega_0 R_0^2$ and ϕ_1 and ϕ_2 represent the small disturbance.

(a) Write down the kinematic conditions at the vortex sheet which link ϕ_1, ϕ_2 and $\xi(\theta, t)$, the radial displacement of the vortex sheet away from its equilibrium position.

(b) Using the time-dependent Bernoulli relation $\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + p\right] = 0$ and an azimuthal wavenumber m (as in $(*)$), show that the dynamical condition (to be evaluated at $r = R_0$) is

$$\sigma(\phi_1 - \phi_2) - \frac{im\Gamma_0}{R_0^2}\phi_2 + \frac{\Gamma_0^2}{R_0^3}\xi = \text{const.}$$

(c) Hence show that for this specific choice of $\Omega(r)$, the growth rate of the perturbation is given by

$$\sigma = \left[-\frac{1}{2}im \pm \frac{1}{2}\sqrt{m^2 - 2m}\right]\Omega_0$$

and so that an instability exists.

2

Consider the following simple model of Rayleigh-Benard convection

$$\frac{\partial w}{\partial t} + \frac{\partial^3(\frac{1}{2}w^2)}{\partial z^3} = \Delta^3 w - R\Delta_1 w$$

for $-\infty < x, y < \infty$ and $0 < z < 1$ with

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \qquad \text{at } z = 0, 1.$$

Here R represents the Rayleigh number and $\Delta_1 := \Delta - \partial^2/\partial z^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

(i) Taking normal modes of the form $w' = W(z)f(x,y)e^{st}$ to solve the linearized problem, show that

$$W = W_j = A_j \sin j\pi z \quad \text{for } j = 1, 2, \ldots \qquad \text{and} \quad \Delta_1 f + a^2 f = 0,$$

where a is any real number and give an expression for $s = s(a, R, j)$. For a given value of a, show that the $j = 1$ mode is unstable first as R increases and find this critical value $R_c(a)$.

(ii) Taking the 'roll cell' $w_1 = A\cos ax \sin \pi z$ as the fundamental in the weakly nonlinear problem, deduce that its Landau equation is

$$\frac{dA}{d\tau} = a^2(R - R_c)A - \frac{1}{4}\pi^3(2d_0 + d_1)A^3 \qquad (*)$$

where τ is a suitably defined slow time variable,

$$d_0 = \frac{1}{64\pi^3} \quad \text{and} \quad d_1 = \frac{\pi^3}{60(\pi^2 + a^2)^3}.$$

(iii) Sketch a bifurcation diagram for the equation $(*)$ in the $(R - R_c, A)$ plane clearly indicating the behaviour of $A(\tau)$ for values of $R < R_c$ and $R > R_c$ and indicating all equilibrium solutions.

3

(i) State the condition for a matrix $L \in \mathbb{C}^{n\times n}$ to be non-normal and demonstrate how non-normality can lead to transient energy growth in the system $\dot{\mathbf{x}} = L\mathbf{x}$ by showing it in a simple $n = 2$ example.

(ii) Show that if L is non-normal and diagonalisable then there is still a norm in which no growth is ever possible if all the eigenvalues indicate damped behaviour.

(iii) Now consider the matrix

$$L = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are both real and negative.

(a) Show that the solution of $\dot{\mathbf{x}} = L\mathbf{x}$ is $\mathbf{x}(t) = A\mathbf{x}(0)$ where

$$A = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} & e^{\lambda_2 t} \end{bmatrix}$$

(b) Derive a quadratic equation whose roots are the maximum and minimum growth possible after a time t in the standard norm $|\mathbf{x}|^2$.

(c) Take the limit $\lambda_1 \to \lambda_2 = \lambda$ of the quadratic first to get a simpler expression for the maximum growth at a given t.

(d) Now take your result in (c) and consider the limit $\lambda \to 0$. Deduce the optimal time t_m at which the growth is maximal across all t. Give this maximal value of the growth at t_m.

END OF PAPER