UNIVERSITY OF CAMBRIDGE

MAMA/329, NST3AS/329, MAAS/329

MAT3

MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 1:30 pm to 4:30 pm

PAPER 329

SLOW VISCOUS FLOW

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A thin sheet of very viscous fluid flows with a characteristic velocity U between two rigid surfaces with a characteristic gap width H that varies on a length scale $L \gg H$. Show by means of scaling arguments that the characteristic shear stress τ and characteristic pressure variations P satisfy $\tau/P \ll 1$.

The thin gap between two rigid spheres of radius a and $a + \Delta$, with $0 < \Delta \ll a$, is filled with fluid of viscosity μ. The outer sphere is stationary, and motion of the inner sphere causes flow in the gap, which is described by the lubrication equations. The centre of the inner sphere is a distance $\lambda\Delta$ (with $-1 < \lambda < 1$) directly below the centre of the outer sphere. Show by means of a sketch, or otherwise, that the gap width $h(\theta)$ between the spheres is approximately given by

$$h = (1 - \lambda \cos\theta)\Delta,$$

where θ is the angle to the downward vertical.

(i) Show that when the inner sphere moves downwards with speed $V = (d\lambda/dt)\Delta$ the local width-integrated flux is given by $q = \frac{1}{2}Va\sin\theta$. Deduce that the fluid pressure is given by

$$p(\theta) = \frac{3\mu a^2 V}{\lambda\Delta^3(1 - \lambda\cos\theta)^2} + \text{const.}$$

and calculate the vertical force F acting on the inner sphere.

[*You may assume that* $\displaystyle\int_{-b}^{b} \frac{t\,dt}{(1-t)^2} = \frac{2b}{(1-b)^2} + \ln\left(\frac{1-b}{1+b}\right)$ for $|b| < 1$.]

(ii) Calculate also the couple G acting on the inner sphere when it rotates with angular velocity Ω about the vertical axis.

[*You may assume that* $\displaystyle\int_{-b}^{b} \frac{(c^2 - t^2)\,dt}{1-t} = 2b + (1-c^2)\ln\left(\frac{1-b}{1+b}\right)$ for $|b| < 1$.]

(iii) The inner sphere now rotates with angular velocity Ω' about a horizontal axis, with its centre held stationary. The motion results from a horizontal couple G' applied to the inner sphere about its centre, together with, if $\lambda \neq 0$, a horizontal force F' to hold the centre stationary. Explain why no vertical force is necessary.

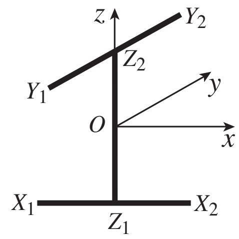
Let $\lambda = 1 - \epsilon$, where $0 < \epsilon \ll 1$. Show that $h = O(\epsilon\Delta)$ in a small circular region on the inner sphere of radius $O(\epsilon^{1/2}a)$.

Use scaling arguments to show that the contributions to the couple G' from the region where $h = O(\Delta)$ and from the region where $h = O(\epsilon\Delta)$ are both of magnitude $O(\mu\Omega' a^4/\Delta)$.

Estimate similarly the magnitude of the contributions to the horizontal force F' from these regions.

2 (a) State the reciprocal theorem for two Stokes flows in which the body forces are zero. Prove that the resistance matrix, giving the force $\mathbf{F}$ and couple $\mathbf{G}$ exerted by a rigid body moving with velocity $\mathbf{U}$ and angular velocity $\boldsymbol{\Omega}$ through unbounded viscous fluid otherwise at rest, is symmetric.

(b) Such a rigid body comprises three equal straight rods, extending from $X_1 = (-1, 0, -1)L$ to $X_2 = (1, 0, -1)L$, from $Y_1 = (0, -1, 1)L$ to $Y_2 = (0, 1, 1)L$ and from $Z_1 = (0, 0, -1)L$ to $Z_2 = (0, 0, 1)L$, and joined at their intersections Z_1 and Z_2. Rotations and couples are defined to be about the mid-point $O = (0, 0, 0)$.



Use symmetry arguments to show that if $\mathbf{F} = (0, 0, F)$ and $\mathbf{G} = \mathbf{0}$ then the body translates in the z-direction and does not rotate. Write down which components of $\mathbf{U}$ and $\boldsymbol{\Omega}$ are nonzero if, instead, $\mathbf{F} = (F, 0, 0)$ and $\mathbf{G} = \mathbf{0}$.

(c) The force per unit length exerted by the rods is given by the usual slender-body formula

$$\mathbf{f}(\mathbf{X}) = C(\mathbf{I} - \tfrac{1}{2}\mathbf{X}'\mathbf{X}') \cdot \dot{\mathbf{X}},$$

where $C = 4\pi\mu/\ln(1/\epsilon)$ is a constant and $\mathbf{X}$ is the position along the rod. Use simple arithmetic to calculate the nonzero Cartesian components of $\mathbf{F}$ and $\mathbf{G}$ when the body translates with speed U (and $\boldsymbol{\Omega} = \mathbf{0}$) in each of the three coordinate directions. Deduce that $\mathbf{G}$ is proportional to $(U_y, U_x, 0)$ for any velocity $\mathbf{U} = (U_x, U_y, U_z)$ when $\boldsymbol{\Omega} = \mathbf{0}$.

When $\boldsymbol{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ and $\mathbf{U} = \mathbf{0}$ the couple exerted by the body is $\mathbf{G} = CL^3(\frac{13}{3}\Omega_x, \frac{13}{3}\Omega_y, \frac{4}{3}\Omega_z)$. Combine this information with results from parts (a) and (c) to write down the full 6×6 resistance matrix.

(d) Show that if $\mathbf{F} = CL\,\mathbf{k}$ and $\mathbf{G} = \mathbf{0}$ then

$$\mathbf{U} = (\alpha k_x, \alpha k_y, \tfrac{1}{5}k_z) \quad \text{and} \quad L\boldsymbol{\Omega} = (-\gamma k_y, -\gamma k_x, 0),$$

where the numerical coefficients α and γ are to be determined.

(e) The coordinates and components of vectors above are defined with respect to the axes fixed in the body, which will be rotating in space if $\boldsymbol{\Omega} \neq \mathbf{0}$. The body is actually falling under gravity so that $\mathbf{F}$ is fixed in space (vertically downwards) and $\mathbf{G} = \mathbf{0}$. Explain why the components of $\mathbf{k}$, still defined with respect to axes fixed in the body, obey

$$\frac{d}{dt}(k_x, k_y, k_z) = \frac{\gamma}{L}\big(k_x k_z,\ -k_y k_z,\ k_y^2 - k_x^2\big).$$

Show that if $\boldsymbol{\Omega}$ is constant and nonzero then $\mathbf{k}$ must be parallel to $(1, 1, 0)$ or $(1, -1, 0)$. Describe with the aid of a sketch the orientation and motion of the body in physical space if $\mathbf{k} = (1, 1, 0)$. [Detailed calculation is not required.]

If, instead, $\mathbf{k} = (1, 0, 0)$ at $t = 0$, describe briefly the subsequent motion of the body in physical space including its orientation as $t \to \infty$. [Detailed calculation is not required.]

3 (a) An infinite fluid cylinder of radius a_0, viscosity μ and (uniform) surface tension γ has perturbations to its shape given by $r = a(z,t)$ in cylindrical polar coordinates, where $a = a_0 + \eta(z,t)$, $|\eta| \ll a_0$ and $|\partial\eta/\partial z| \ll 1$. Calculate the linearised perturbation to the curvature. State the linearised boundary conditions on the resulting flow.

State the Papkovich–Neuber representation for Stokes flow.

For $\eta \propto e^{ikz+st}$ (with real part understood), the flow is given by potentials $\mathbf{\Phi} = \big(P\,\mathrm{I}_1(kr)e^{ikz+st}, 0, 0\big)$ and $\chi = Qk^{-1}\,\mathrm{I}_0(kr)e^{ikz+st}$, where P and Q are constants and $\mathrm{I}_0(x)$ and $\mathrm{I}_1(x)$ are modified Bessel functions. Calculate the corresponding velocity components u and w and the tangential stress σ_{rz}. Deduce that P and Q satisfy

$$Px\,\frac{d\mathrm{I}_1}{dx} + Q\,\frac{d\mathrm{I}_0}{dx} = 0 \quad \text{when} \quad x = ka\,.$$

(b) Now consider the case of a cylinder covered with a concentration $C(z,t)$ of insoluble surfactants. The (nonuniform) surface tension is given by

$$\gamma(C) = \gamma_0 - AC'\,,$$

where γ_0 and A are positive constants, $C' = C - C_0$, $|C'| \ll C_0$, and C_0 is the uniform concentration if $a = a_0$. Diffusion of surfactant is negligible.

Assume that the long-wavelength limit is extensional flow (like the case $A = 0$), with $w(z,t)$ independent of r and $u(r,z,t)$ given by mass conservation. At leading order the nonlinear evolution equations can be written

$$\frac{\partial}{\partial z}\Big\{\pi a^2\Big(-\frac{\gamma}{a} + 3\mu\frac{\partial w}{\partial z}\Big) + 2\pi a\gamma\Big\} = 0, \qquad \frac{Da^2}{Dt} = -a^2\frac{\partial w}{\partial z}, \qquad \frac{DC}{Dt} = -C\Big(\frac{\partial w}{\partial z} + \frac{u}{a}\Big).$$

Describe the physical interpretation of the various parts of the first and third equations.

Obtain linearised equations for small perturbations to a uniform state with $a = a_0$, $C = C_0$, $\gamma = \gamma_0$ and $w = 0$, such that there is no perturbation to the net axial force $\pi a_0\gamma_0$. Deduce that $a_0C'(z,t) = C_0\eta(z,t) + \psi(z)$, where ψ is determined by the initial conditions.

Show further that

$$\frac{\partial\eta}{\partial t} = s\eta - \frac{A\psi(z)}{6\pi\mu a_0}\,,$$

where the constant s is to be determined.

Initially $\eta = 0$ and $C' > 0$. Describe *physically* what happens to η and C' (and why) in each of the cases (i) $A < \gamma_0/C_0$ and (ii) $A > \gamma_0/C_0$.

(c) Now consider the case $A > \gamma_0/C_0$ with initial conditions in which $a\gamma = a_0\gamma_0$, but a and γ are nonuniform. According to the extensional-flow equations in part (b), these conditions would result in there being no flow ($w = 0$). By considering the capillary pressure, explain briefly why that cannot actually be true.

Assume that the flow should instead be described by lubrication theory, with an axial Poiseuille flow driven by the pressure gradient. Use simple scaling arguments to show that $\partial\eta/\partial t \sim (\gamma_0 a_0/\mu L^2)\eta$, where L is the axial length scale of variation.

END OF PAPER