UNIVERSITY OF CAMBRIDGE

MAMA/327, NST3AS/327, MAAS/327

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 9:00 am to 11:00 am

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)

(i) Define the *Schwartz space* $\mathcal{S}(\mathbf{R})$ and the space of *tempered distributions* $\mathcal{S}'(\mathbf{R})$. Define the *Fourier transform* on each of these spaces.

(ii) Show that if $f \in L^1(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$ then the Fourier transform $\hat{f} \in \mathcal{S}'(\mathbf{R})$ can be identified with a bounded, continuous function.

(b) Let $u \in \mathcal{S}'(\mathbf{R})$ be defined by the function

$$u(x) = \begin{cases} 1/x, & x > 1 \\ 0, & x \leqslant 1. \end{cases}$$

(i) Show that for $\varphi \in \mathcal{S}(\mathbf{R})$

$$\langle \hat{u}, \varphi \rangle = \int \varphi'(\lambda) v(\lambda)\, \mathrm{d}\lambda \quad \text{where} \quad v(\lambda) = -\mathrm{i} \int_1^\infty \frac{e^{-\mathrm{i}\lambda x}}{x^2}\, \mathrm{d}x.$$

(ii) Show that $v \in C(\mathbf{R}) \cap C^\infty(\mathbf{R} \setminus \{0\})$.

(iii) Establish the following identity for v when $\lambda \in (0,1)$

$$v(\lambda) = \lambda \log|\lambda| - \mathrm{i}e^{-\mathrm{i}\lambda} - \lambda \left[\int_\lambda^1 \frac{(e^{-\mathrm{i}x} - 1)}{x}\, \mathrm{d}x + \int_1^\infty \frac{e^{-\mathrm{i}x}}{x}\, \mathrm{d}x \right]$$

and by taking the complex conjugate of a suitable function, or otherwise, find a similar identity valid for $\lambda \in (-1, 0)$.

(iv) Deduce that there exist smooth functions $f_\pm \in C^\infty(\mathbf{R}^\pm)$ such that

$$\hat{u}(\lambda) + \log|\lambda| = f_\pm(\lambda) \qquad \pm\,\lambda > 0.$$

(v) Show that

$$\lim_{\lambda \downarrow 0} \left[f_+(\lambda) - f_-(-\lambda) \right] = -\mathrm{i}\pi.$$

[You may assume $\int_0^\infty (\sin x / x)\, \mathrm{d}x = \frac{\pi}{2}$.]

(vi) Suppose that $w \in C(\mathbf{R})$ and $|w(x) - c_\pm/x| \lesssim 1/x^2$ as $x \to \pm\infty$ for two constants $c_\pm$. Show that the limits

$$\lim_{\lambda \downarrow 0} \left(\hat{w}(\lambda) + (c_+ - c_-) \log|\lambda| \right), \quad \lim_{\lambda \uparrow 0} \left(\hat{w}(\lambda) + (c_+ - c_-) \log|\lambda| \right)$$

exist and compute their difference.

[*Hint:* if u is the distribution considered in parts (i)-(v) consider

$$w(x) - \Big[\alpha\, u(x) + \beta\, u(-x) \Big]$$

for some constants α, β.]

2 Define the space of *test functions* $\mathcal{D}(\mathbf{R}^n)$ and the space of *distributions* $\mathcal{D}'(\mathbf{R}^n)$, defining the notion of *convergence* on each. Your definition of $\mathcal{D}'(\mathbf{R}^n)$ should involve a semi-norm estimate.

Show that a linear form $u : \mathcal{D}(\mathbf{R}^n) \to \mathbf{C}$ defines an element of $\mathcal{D}'(\mathbf{R}^n)$ if and only if $\langle u, \varphi_m \rangle \to 0$ for each sequence of test functions $\{\varphi_m\}_{m \geqslant 1}$ that converge to zero in $\mathcal{D}(\mathbf{R}^n)$.

Let $h \in \mathbf{R}^n$ and α be a multi-index. For $u \in \mathcal{D}'(\mathbf{R}^n)$ define *translation* $u \mapsto \tau_h u$ and *differentiation* $u \mapsto D^\alpha u$.

Let e_i denote the unit vector along the ith coordinate axis. Show that for $u \in \mathcal{D}'(\mathbf{R}^n)$, $\tau_{te_i} u = u$ for all $t \in \mathbf{R}$ if and only if $\partial u / \partial x_i = 0$ in $\mathcal{D}'(\mathbf{R}^n)$.

For $f, g \in L^1_{\text{loc}}(\mathbf{R})$ define the distribution $u \in \mathcal{D}'(\mathbf{R}^2)$ by the function

$$u(x, y) = f(x - y) + g(x + y).$$

Show that $u_{xx} - u_{yy} = 0$ in $\mathcal{D}'(\mathbf{R}^2)$.

3 (a) Let $X \subset \mathbf{R}^n$ be open. What does it mean for $\Phi : X \times \mathbf{R}^k \to \mathbf{R}$ to be a *phase function*? Define the space $\text{Sym}(X, \mathbf{R}^k; N)$ and show that:

(i) If $a \in \text{Sym}(X, \mathbf{R}^k; N)$ then $D_x^\alpha D_\theta^\beta a \in \text{Sym}(X, \mathbf{R}^k; N - |\beta|)$.

(ii) If $a_i \in \text{Sym}(X, \mathbf{R}^k; N_i)$ for $i = 1, 2$ then $a_1 a_2 \in \text{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

(iii) If $b \in C^\infty(X \times \mathbf{R}^k)$ is positively homogeneous of degree M in θ for $|\theta|$ sufficiently large, then $b \in \text{Sym}(X, \mathbf{R}^k; M)$.

For Φ a phase function and $a \in \text{Sym}(X, \mathbf{R}^k; N)$ define

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x, \theta) \, \mathrm{d}\theta$$

in terms of a linear map from $\mathcal{D}(X)$ to $\mathbf{C}$ and show that $I_\Phi(a) \in \mathcal{D}'(X)$.

(b) If $(x, \theta) \in \mathbf{R} \times \mathbf{R}$, show that

$$\int e^{\mathrm{i}x\sqrt{\theta^2+1}} \, \mathrm{d}\theta$$

defines an element of $\mathcal{D}'(\mathbf{R})$.

[You may assume that for $\theta > 2$ and $\alpha, \beta = 0, 1, 2, \ldots$

$$\left| D_\theta^\beta \left(\frac{1}{\sqrt{\theta^2 + 1} + \theta} \right)^\alpha \right| \lesssim_{\alpha,\beta} \langle \theta \rangle^{-\beta} .]$$

END OF PAPER