UNIVERSITY OF CAMBRIDGE

MAMA/321, NST3AS/321, MAAS/321

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Viscous and magnetic accretion

An axisymmetric viscous accretion disc is threaded by a large-scale magnetic field **B** anchored in the surrounding interstellar medium (taken to be at infinity). In cylindrical coordinates, the disc's evolution is governed by

$$\partial_t\rho + \frac{1}{r}\partial_r(r\rho u_r) + \partial_z(\rho u_z) = 0,$$

$$\rho\left[\partial_t u_\phi + \frac{1}{r}u_r\partial_r(ru_\phi) + u_z\partial_z u_\phi\right] = \frac{1}{r^2}\partial_r(r^2\Pi_{r\phi}) + \partial_z\Pi_{\phi z} + \frac{1}{r^2}\partial_r\left(r^2\frac{B_rB_\phi}{4\pi}\right) + \partial_z\left(\frac{B_\phi B_z}{4\pi}\right),$$

where $\mathbf{\Pi}$ is the viscous stress and the other symbols take their usual meanings.

(a) Assume that $\rho\mathbf{u}$, $\mathbf{\Pi}$, and B_r go to zero as $|z|\to\infty$, but that $B_\phi B_z$ is non-zero at infinity. You may also take $u_\phi = r\Omega(r)$. Hence derive the following 1D advection-diffusion equation:

$$\partial_t\Sigma = \frac{1}{2\pi r}\partial_r\left[\left(\frac{dr}{dh}\right)(\partial_r\mathcal{G} - r\mathcal{T})\right],$$

where Σ is surface density, h is specific angular momentum, $\mathcal{G}$ is the total internal viscous-magnetic torque, and $\mathcal{T}$ is the total magnetic torque at $|z|\to\infty$, expressions for which you must provide.

(b) Let $\mathcal{G}=0$ and set $\mathcal{T} = 2\pi v_m(dh/dr)\Sigma$, where v_m is a negative drift speed, potentially a function of the other variables.

(i) If the disc is receiving mass at its outer edge at a rate $\dot{M}$, find the expression that $\Sigma(r)$ must satisfy for the disc to be in steady state.

(ii) Consider the special case that v_m is a constant. At time $t=0$, $\Sigma=\Sigma_0(r)$, where $\Sigma_0(r)$ describes an initial state localised to $r=r_0$. By considering the mass per unit radius $M=2\pi r\Sigma$, or otherwise, qualitatively describe the disc's evolution and estimate the time it takes for the disc to accrete onto the central star (taken to be at $r=0$). Explain the evolution of the system's angular momentum.

(iii) Suppose that $v_m = v_m(M)$, with $|d(Mv_m)/dM|$ an increasing function of M. Via a diagram or otherwise, qualitatively describe the evolution of a localised initial condition.

(c) Assume now that $\mathcal{G} = 3\pi h\nu\Sigma$, $\mathcal{T} = 2\pi\beta r^{-1/2}(dh/dr)\Sigma$, with ν and β constants ($\beta\leqslant 0$), and that the disc is Keplerian. Suppose it is in steady state, receiving mass at a rate $\dot{M}$ at its outer edge, and $\mathcal{G}=0$ at its inner edge $r=r_{\text{in}}$.

(i) When $\beta=0$, find the steady state profile of $\nu\Sigma(r)$.

(ii) When $\beta\neq 0$, show that

$$\nu\Sigma = \frac{\dot{M}}{3\pi\lambda}x^{-1/2}\left[1-\exp(\lambda f(x))\right], \qquad (\dagger)$$

where $\lambda = -2\beta\sqrt{r_{\text{in}}}/(3\nu)$, $x = r/r_{\text{in}}$, and $f(x)$ is a function you need to find.

(iii) Show that, in two distinct limits of λ, ($\dagger$) agrees with your answers to part (b)(i) and part (c)(i). Estimate the radii where $\mathcal{T}$ dominates in ($\dagger$). At which radii does $\mathcal{G}$ dominate?

2 The radial drift of dust and the streaming instability

(a) A non-turbulent gaseous disc is in a steady axisymmetric state, orbiting a star of mass $M_\star$ with orbital frequency Ω. It satisfies hydrostatic balance:

$$-r\Omega^2\mathbf{e}_r = -\frac{1}{\rho}\nabla P - \nabla\Phi,$$

where ρ and P are the disc's density and pressure, $\Phi = -GM_\star/\sqrt{r^2+z^2}$, and cylindrical coordinates are employed, with $\nabla f = \mathbf{e}_r\partial_r f + \mathbf{e}_z\partial_z f$ for any axisymmetric function f. You may assume the disc is thin.

(i) By expanding Φ, obtain an order of magnitude estimate for the disc semi-thickness H in terms of the sound speed and Ω.

(ii) Suppose the disc is composed of a perfect gas with entropy $s = \ln(\rho^{-\gamma}P)$ (in suitable units) and adiabatic index γ ($\neq 1$). If $s = s(r)$, show that

$$\rho = \rho_0(r)\left(1 - \frac{z^2}{H(r)^2}\right)^m,$$

where ρ_0 is the midplane density, and you need to find expressions for H and m.

(iii) If the midplane pressure is a decreasing function of r, show that the midplane rotation is sub-Keplerian with relative difference in orbital frequency $\sim (H/r)^2$.

(b) The gaseous disc is permeated with dust, which may be treated as a pressure-less fluid of test particles. The dust can be described with a shearing sheet located at radius r_0, moving at the local Keplerian frequency Ω_K. Its governing equations are

$$\partial_t\sigma + \mathbf{u}\cdot\nabla\sigma = -\sigma\nabla\cdot\mathbf{u}, \qquad \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -2\Omega_K\mathbf{e}_z\times\mathbf{u} + 3\Omega_K^2 x\mathbf{e}_x - \frac{1}{\tau}(\mathbf{u}-\mathbf{v}),$$

where σ and $\mathbf{u}$ are the dust density and velocity, τ is a constant, and $\mathbf{v}$ the gas velocity.

(i) In the shearing sheet, the background gas equilibrium of part (a) may be represented as $\mathbf{v} = \mathbf{v}_0 = [V - (3/2)\Omega_K x]\mathbf{e}_y$, where V is a constant. Explain the origin and sign of V.

(ii) Calculate the dust equilibrium, $\sigma = \sigma_0$, $\mathbf{u} = \mathbf{u}_0 = -(3/2)\Omega_K x\mathbf{e}_y + \Delta\mathbf{u}$, where σ_0 is a given constant and $\Delta\mathbf{u}$ is a constant vector you need to find. Give a physical explanation for why the dust drifts radially. Show that this radial drift is maximised when $\tau\Omega_K = 1$.

(c) Suppose an axisymmetric wave is travelling through the gas described in part (b), so that $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}-\mathrm{i}\omega t}$, where $\mathbf{v}_1$ is constant, $\mathbf{k} = k_x\mathbf{e}_x + k_z\mathbf{e}_z$ is a real wavevector, and ω is the real wave frequency. The gas wave causes a small perturbation to the dust equilibrium of part (b), so that $\sigma = \sigma_0 + \sigma'(t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$, and $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'(t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$.

(i) Write down the linearised equations governing the small amplitudes $\sigma'(t)$ and $\mathbf{u}'(t)$, in terms of the operators $\mathcal{D} = \partial_t + \mathrm{i}(\mathbf{k}\cdot\Delta\mathbf{u})$ and $\mathcal{D}_\tau = \mathcal{D} + 1/\tau$. Hence show that

$$\mathcal{D}_\tau(\mathcal{D}_\tau^2 + \Omega_K^2)\mathcal{D}\sigma' = F\mathrm{e}^{-\mathrm{i}\omega t},$$

where F is a constant you need not evaluate.

(ii) Show that $|\sigma'|$ undergoes algebraic growth in time when a certain condition is satisfied, which you should state.

3 Vortensity in the shearing sheet

The equations governing non-self-gravitating isothermal gas in a shearing sheet model of a razor thin disc are

$$D_t\Sigma = -\Sigma\nabla\cdot\mathbf{u}, \qquad D_t\mathbf{u} = -2\Omega\,\mathbf{e}_z\times\mathbf{u} - \frac{1}{\Sigma}\nabla P - \nabla\Phi_t,$$

where $D_t = \partial_t + \mathbf{u}\cdot\nabla$, Σ is surface density, $\mathbf{u}$ is velocity, $\Phi_t = -\frac{3}{2}\Omega^2x^2$ is the tidal potential, Ω is the orbital frequency of the sheet, and $P = c_s^2\Sigma$ is the pressure, with c_s a constant.

(a) The vortensity is defined to be $\zeta = \Sigma^{-1}[2\Omega + (\nabla\times\mathbf{u})\cdot\mathbf{e}_z]$. Show that $D_t\zeta = 0$.

[You may need the vector identities:

$\mathbf{a}\cdot\nabla\mathbf{a} = (\nabla\times\mathbf{a})\times\mathbf{a} + \frac{1}{2}\nabla|\mathbf{a}|^2, \qquad \nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a}.$ *]*

(b) Suppose the disc is in a steady axisymmetric equilibrium whereby $u_x = 0$, $u_y = u_y(x)$, and $\Sigma = \Sigma_0\exp[\sigma(x)]$, where Σ_0 is a constant, and the dimensionless function σ goes to 0 as $|x|\to\infty$.

(i) Write down the relation σ and u_y must satisfy for equilibrium to hold.

(ii) If $\zeta = \zeta(x)$ is known, show that σ must satisfy the following ODE

$$\frac{d^2\sigma}{d\xi^2} - q\mathrm{e}^{\sigma} + 1 = 0, \qquad (\dagger)$$

where $\xi = (\Omega/c_s)x$, and $q = [2\Sigma_0/\Omega]\zeta$.

(iii) Assume that σ is small, so that ($\dagger$) may be linearised in σ, and that

$$q = \begin{cases} 1, & \text{for } |\xi| > d, \\ 0, & \text{for } |\xi| < d, \end{cases}$$

where d is a constant. Hence solve ($\dagger$) for σ and provide a rough plot of Σ.

(c) Perturb the equilibrium $\Sigma = \Sigma_0$, $\mathbf{u} = -\frac{3}{2}\Omega x\,\mathbf{e}_y$ with small disturbances Σ' and $\mathbf{u}'$.

(i) Write down the components of the linearised, perturbed equation of motion.

(ii) Assume the perturbations are of the form

$$\mathbf{u}' = \widetilde{\mathbf{u}}(t)\exp[\mathrm{i}k_x(t)x + \mathrm{i}k_y y], \qquad \Sigma' = \widetilde{\Sigma}(t)\exp[\mathrm{i}k_x(t)x + \mathrm{i}k_y y],$$

where k_x is a time-dependent wavenumber and k_y is constant.

Find and solve the equation k_x must satisfy for the perturbation equations to be consistent. Describe the evolution of the perturbation's morphology.

(iii) The linearised vortensity perturbation is $\zeta' = \widetilde{\zeta}(t)\exp[\mathrm{i}k_x(t)x + \mathrm{i}k_y y]$. Show that $\widetilde{\zeta}$ is a constant. By setting $\widetilde{\zeta} = 0$, express $\widetilde{\Sigma}$ in terms of the components of $\widetilde{\mathbf{u}}$. Hence obtain

$$\frac{d\widetilde{\mathbf{u}}}{dt} = \mathbf{A}(t)\widetilde{\mathbf{u}}, \qquad (\star)$$

where $\mathbf{A}(t)$ is a matrix function you need to determine.

(iv) Set $k_y = 0$ and solve ($\star$), writing down the resulting dispersion relation. What type of disturbance does it describe?

END OF PAPER