UNIVERSITY OF CAMBRIDGE

MAMA/318, NST3AS/318, MAAS/318

MAT3
MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 9:00 am to 11:00 am

PAPER 318

APPROXIMATION THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For a 2π-periodic function $f \in C(\mathbb{T})$, let $s_n(f)$ be its partial Fourier sum of degree n, and let $\sigma_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} s_k(f)$ be its Fejer sum of degree $n-1$. Further, let $\|f\|$ be the max-norm of f on $\mathbb{T} := [-\pi, \pi]$.

(a) From the integral representation

$$s_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-t)f(t)\,dt, \qquad D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$

and the relation between s_n and σ_n, derive the following expression for the Fejer sum σ_n and the Fejer kernel F_n

$$\sigma_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} F_n(x-t)f(t)\,dt\,, \qquad F_n(x) = \frac{1}{2n}\frac{\sin^2\frac{n}{2}x}{\sin^2\frac{1}{2}x}\,.$$

Hence prove that $\frac{1}{\pi}\int_{\mathbb{T}} |F_n(t)|\,dt = 1$.

(b) Prove that, if $f \in \operatorname{Lip}\alpha$ for some $0 < \alpha \leqslant 1$, i.e. $|f(x) - f(y)| \leqslant M|x-y|^\alpha$, then there exist constants c_α and c_1 such that

$$\|\sigma_n(f) - f\| \leqslant \begin{cases} c_\alpha n^{-\alpha}\,, & 0 < \alpha < 1\,, \\ c_1\frac{\ln n}{n}\,, & \alpha = 1 \end{cases}$$

(c) Using the inequality

$$\int_{I_k} \frac{\sin^2\frac{n}{2}t}{t}\,dt > \frac{c}{k}\,, \qquad I_k = (\tfrac{2\pi(k-1)}{n}, \tfrac{2\pi k}{n}), \qquad k = 1,\dots,n', \quad n' = \lfloor\tfrac{n}{2}\rfloor\,,$$

where c is some constant, prove that, for the 2π-periodic function $f_0(x) = |x|$, $x \in [-\pi,\pi]$, we have

$$|\sigma_n(f_0,x) - f_0(x)|_{x=0} \;\geqslant\; c_1'\frac{\ln n}{n}\,.$$

Is it possible to have the Jackson-type estimate $\|\sigma_n(f) - f\| \leqslant c\,\omega(f,\frac{1}{n})$ for approximation of continuous functions by the Fejer sums? (Here $\omega(f,\delta)$ is the modulus of continuity of f.)

2

For a function $f \in C[-1,1]$ let $E_n(f)$ be the value of the best approximation of f in the uniform norm from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n.

(a) State the Chebyshev alternation theorem for the element of best approximation $p^* \in \mathcal{P}_n$ such that $E_n(f) = \|f - p^*\|$.

(b) Given $f \in C[-1,1]$, let $p \in \mathcal{P}_n$ be a polynomial such that, for some $n+2$ points $t_1 < t_2 < \cdots < t_{n+2}$ in $[-1,1]$, we have

$$f(t_i) - p(t_i) = (-1)^i a_i, \qquad a_i > 0.$$

Prove that

$$E_n(f) \geqslant \min_{1 \leqslant i \leqslant n+2} a_i \,.$$

(c) It is clear that, for any f, we have $E_{n-1}(f) \geqslant E_n(f)$. For every n give an example of function $f = f_n$ such that $E_{n-1}(f) = E_n(f)$. Show that if $f^{(n)}(x) > 0$ on $[-1,1]$, then we have the strict inequality $E_{n-1}(f) > E_n(f)$.

(d) Let $T_n(x) = \cos n \arccos x$ be the Chebyshev polynomial of degree n, and let

$$f_0(x) = \sum_{k=0}^{\infty} a_k T_{3^k}(x), \quad \text{where} \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty, \quad x \in [-1,1].$$

Prove that, for every n, the polynomial p_n of best approximation to f_0 in $C[-1,1]$ is given by a partial sum of the series above, and for each n determine the value $E_n(f_0)$ of best approximation in terms of a_k.

3

Let $\Delta = (t_j)_{j=1}^{n+k}$ be a strictly increasing knot sequence, and let $\mathcal{S}_k(\Delta)$ be the space of splines of degree $k-1$ spanned by the B-splines $(N_j)_{j=1}^{n}$.

Let $\boldsymbol{x} = (x_i)_{i=1}^{n}$ be another strictly increasing point sequence, and denote by $P_{\boldsymbol{x}} : C[a,b] \to \mathcal{S}_k(\Delta)$ the map which associates with any $f \in C[a,b]$ the spline $P_{\boldsymbol{x}}(f)$ from $\mathcal{S}_k$ which interpolates f at (x_i) (if it exists). Further, let $A_{\boldsymbol{x}}$ be the matrix $(N_j(x_i))_{i,j=1}^{n}$.

(a) Prove that if $A_{\boldsymbol{x}}$ is invertible then the following conditions are fulfilled

$$N_i(x_i) > 0\,, \qquad i = 1, \ldots, n\,.$$

State the Schoenberg–Whitney theorem about invertibility of $A_{\boldsymbol{x}}$.

(b) Prove that, if $A_{\boldsymbol{x}}^{-1}$ exists, then

$$\|P_{\boldsymbol{x}}\|_{L_\infty} \leqslant \|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty}\,.$$

(c) Consider the case of cubic interpolating splines on the uniform knot-sequence $(t_1, t_2, \ldots, t_{n+4}) = (1, 2, \ldots, n+4)$ with the interpolating points

$$x_i = t_{i+2} = i + 2, \quad i = 1, \ldots, n\,.$$

Using the recurrence relation for B-splines or otherwise, determine the entries of the matrix $A_{\boldsymbol{x}}$, and hence estimate the norm $\|A_{\boldsymbol{x}}^{-1}\|_{\ell_\infty}$. (You may use any appropriate theorem on the inverse of certain matrices if correctly stated.) Thus show that $\|P_{\boldsymbol{x}}\|_{L_\infty} \leqslant 3$.

4

Given $\Delta = (t_i)_{i=1}^{n+k}$, let ω_i and ψ_i be the polynomials in $\mathcal{P}_{k-1}$ defined as

$$\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1}), \qquad \psi_i(x) := \frac{1}{(k-1)!}\omega_i(x)\,,$$

and let $(N_i)_{i=1}^{n}$ be the corresponding B-spline sequence.

(a) From the Marsden identity

$$(x-t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k \leqslant t \leqslant t_{n+1}, \quad \forall x \in \mathbb{R}$$

prove that any algebraic polynomial $p \in \mathcal{P}_{k-1}$ has the B-spline expansion

$$p(t) = \sum_{i=1}^{n} \lambda_i(p,x) N_i(t), \quad t \in [t_k, t_{n+1}]\,,$$

and express the functional $\lambda_i(p,x)$ in terms of p, ψ_i and $x \in \mathbb{R}$. Explain briefly why the functionals $\lambda_i(p,x)$ are independent of x.

(b) Prove further that that the functionals (λ_i) are dual to the B-spline basis, i.e.

$$\lambda_i(N_j, \xi_i) = \delta_{ij}\,, \quad \forall \xi_i \in [t_i, t_{i+k}]\,.$$

[*Hint. Consider restriction of* (N_j) *on any subinterval* $[t_\ell, t_{\ell+1}]$ *of* $[t_i, t_{i+k}]$]

5

1) Let $\mathbb{X}$ be an inner product space with the scalar product $(\cdot,\cdot)$ and the norm $\|x\| := (x,x)^{1/2}$, and let $\mathcal{U}_n$ be an n-dimensional subspace.

(a) Prove that $u^* \in \mathcal{U}_n$ is the best approximation to $f \in \mathbb{X}$ from $\mathcal{U}_n$ if and only if

$$(f - u^*, v) = 0 \quad \forall v \in \mathcal{U}_n.$$

(b) Let $P_{\mathcal{U}} : \mathbb{X} \to \mathcal{U}_n$ be the linear operator of orthogonal projection onto $\mathcal{U}_n$ defined as

$$(P_{\mathcal{U}} f, v) = (f, v) \quad \forall v \in \mathcal{U}_n\,.$$

Show that $P_{\mathcal{U}} f$ is the best approximation u^* to f from $\mathcal{U}_n$ and $\|u^*\| \leqslant \|f\|$.

2) Let $\mathbb{X} = L_2[a,b]$ with the usual inner product $(f,g) := \int_a^b f(t)g(t)\,dt$.

Given a knot sequence $\Delta = \{a = x_1 < x_2 < \cdots < x_{n+k} = b\}$, and the values $(\gamma_i)_{i=1}^n$, consider the following minimization problem: find

$$\sigma = \arg\min \left\{ \|f^{(k)}\|_2 : f[x_i, \ldots, x_{i+k}] = \gamma_i\,, \quad i = 1, \ldots, n \right\}, \tag{$*$}$$

where $f[x_i, \ldots, x_{i+k}]$ is the divided difference of f of order k. In other words, among all functions with the given values of n divided differences, find the function σ that has the smallest L_2-norm of its k-th derivative $\sigma^{(k)}$.

(a) Show that if $f \in C^k[a,b]$, then

$$f[x_i, \ldots, x_{i+k}] = \frac{1}{k!} \int_a^b M_i(t) f^{(k)}(t)\,dt,$$

where M_i is the k-order B-spline $M_i(t) = k[x_i, \ldots, x_{i+k}](\cdot - t)_+^{k-1}$.

(b) Let $s \in S_k(\Delta)$ be the spline such that, with (γ_i) as given in $(*)$, we have

$$(M_i, s) = k!\gamma_i, \quad i = 1, \ldots, n\,.$$

Write down a linear system of equations for determining the coefficients $a = (a_j)$ of the B-spline expansion of $s = \sum_{j=1}^n a_j M_j$ in the form $Ga = b$, specifying the matrix G and the right-hand side b.

(c) Prove that solution to $(*)$ is given by σ such that $\sigma^{(k)} = s$, where s is the spline from (b).

END OF PAPER