UNIVERSITY OF CAMBRIDGE

MAMA/317, NST3AS/317, MAAS/317

MAT3

MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 1:30 pm to 4:30 pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Briefly describe why, for a homologous set of stars of radius R and mass M, we expect $P_{\rm c} \propto M\rho_{\rm c}/R$, where $P_{\rm c}$ and $\rho_{\rm c}$ are the central pressure and density. A homologous set of fully radiative stars of uniform composition have opacity obeying Kramers' law $\kappa = \kappa_0 \rho T^{-3.5}$ and generate energy via the pp-chain at a rate given by $\epsilon = \epsilon_0 \rho T^{3.5}$, where κ_0 and ϵ_0 are constants and ρ and T are density and temperature which are related to pressure P by an ideal gas equation of state. Show that the luminosity L and radius R of these stars depend on mass M according to

$$L \propto M^{11/2}$$

and

$$R = \text{const.}$$

Explain why such a set of stars is a useful representation of the lower main sequence including the Sun.

(ii) Given that the Sun has a central temperature of $1.5 \times 10^7\,$K and that the CNO-cycle dominates the pp-chain above $2 \times 10^7\,$K in solar composition material, estimate the maximum mass represented by the above homologous series. Above this mass the energy generation rate obeys $\epsilon = \epsilon_0 \rho T^{11.5}$. In all other respects the stars are similar to the homologous series described above. Obtain relations between L and R, and M for such stars.

Sketch the two sequences in a Hertzsprung–Russell diagram.

How does the energy generation rate depend on composition for the pp-chain and the CNO-cycle?

(iii) Another set of stars, formed in a different environment, have the same mass fraction of hydrogen but a mass fraction of CNO elements 256 times smaller than the Sun. Assuming any changes in opacity and mean molecular weight are negligible, estimate the temperature at which the CNO-cycle dominates in these stars and the corresponding mass at which this occurs.

Sketch, on the same Hertzsprung-Russell diagram as the solar composition stars, the two sequences for these stars.

2 (i) In a plane-parallel grey atmosphere of negligible mass and without any source of energy the optical depth τ is defined by $\mathrm{d}\tau = -\kappa\rho \mathrm{d}z$, where $\kappa(\rho, T)$ is the opacity of stellar material of density ρ at temperature T, z is the height in the atmosphere and $\tau \to 0$ at large z. The equation of radiative transfer can be written as

$$\cos\theta \frac{\mathrm{d}I}{\mathrm{d}\tau} = I - \frac{j}{\kappa}, \qquad (*)$$

where $I(\tau, \theta)$ is the intensity of radiation at optical depth τ at an angle θ to the z-axis and j, the effective emissivity, is isotropic and so given by

$$\frac{j}{\kappa} = \frac{\sigma T^4}{\pi},$$

where σ is the Stefan–Boltzmann constant. Integrate $(*)$ over a sphere to deduce that

$$4\pi \frac{j}{\kappa} = \int_{\text{sphere}} I(\tau, \theta)\, \mathrm{d}\Omega = 4\pi J,$$

where $J(\tau)$ is the mean intensity.

(ii) Show that an intensity of the form

$$I(\tau, \theta) = A(\tau) + C(\tau)\cos\theta$$

satisfies the Eddington closure approximation

$$cP_{\mathrm{r}} = \frac{4}{3}\pi J$$

between radiation pressure $P_{\mathrm{r}}(\tau)$, the speed of light c and J, and is a solution to $(*)$ if

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = C \qquad \text{and} \qquad C = \frac{3F}{4\pi},$$

where F is the radiation flux.

Find $A(\tau)$ and use the definition of effective temperature T_{e} to deduce that

$$T^4 = \frac{3}{4} T_{\mathrm{e}}^4 \left(\tau + \frac{2}{3}\right),$$

and that when $\tau = 0$, $T = T_0 = 2^{-1/4} T_{\mathrm{e}}$.

(iii) In the atmosphere of a red dwarf the opacity obeys

$$\kappa = \kappa_0 P^{\alpha-1} T^{4-4\beta}$$

and radiation pressure is negligible. Show that the pressure P varies with temperature as

$$P^{\alpha} = \frac{2\alpha g}{3\kappa_0 \beta T_0^4}\left(T^{4\beta} - T_0^{4\beta}\right),$$

where g is the surface gravity of the star.

Deduce that an appropriate surface boundary condition, for the stellar interior, is

$$\frac{P\kappa}{g} = \frac{4\alpha}{3\beta}\left(1 - 2^{-\beta}\right),$$

where the local luminosity $L_r = 4\pi\sigma r^2 T^4$.

3

(i) A white dwarf has a helium core mass M, radius R and a thin hydrogen-rich, non-degenerate envelope, that behaves as a perfect gas and in which radiation pressure may be neglected, of mass $M_{\rm env} \ll M$ and thickness $H \ll R$. With $z = r - R$, where r is the distance from the centre of the white dwarf, the surface density $\Sigma(z)$ in the envelope satisfies $\mathrm{d}\Sigma = -\rho \mathrm{d}z$ with $\Sigma = 0$ at $z = H$ and $\Sigma = \Sigma_0 = M_{\rm env}/4\pi R^2$ at $z = 0$. Show that, to a good approximation, in the envelope the pressure $P = \Sigma g$ where $g = GM/R^2$.

Show that approximate equations for the structure of the envelope are

$$\frac{\mathrm{d}F}{\mathrm{d}\Sigma} = -\epsilon \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}\Sigma} = \frac{3\kappa F}{4acT^3},$$

where $F = L_r/4\pi R^2$, L_r is the outward luminosity through a sphere of radius r, $T(r)$ is the temperature, ϵ is the specific energy generation rate, κ is the opacity, a is the radiation constant and c is the speed of light.

(ii) The opacity is of the form $\kappa = \kappa_0 \rho T^{-2}$ and κ_0 is a constant. The white dwarf's luminosity is produced solely by hydrogen burning in the envelope. The hydrogen burns steadily, with $\epsilon = \epsilon_0 \rho T^{15}$ and ϵ_0 is a constant. Set $y = T^7$ and $x = \frac{1}{2}\Sigma^2$ and deduce that the structure equations can be written in the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\omega^2 y^2,$$

where ω^2 is a positive constant, that need not be found explicitly.

Show that appropriate boundary conditions at $x = 0$ are

$$y = 0 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{AL}{4\pi R^2},$$

where A is a constant, that need not be found explicitly, and at $x = \frac{1}{2}\Sigma_0^2$

$$y = T_0^7 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

(iii) Integrate once to find that the temperature at the base of the envelope satisfies

$$T_0 \propto \left(\frac{L}{4\pi R^2}\right)^{2/21}.$$

Integrate again, noting that

$$\int_0^1 \frac{\mathrm{d}\eta}{\sqrt{1-\eta^3}}$$

is a constant to deduce that

$$\frac{L}{4\pi R^2} \propto \left(\frac{M_{\rm env}}{4\pi R^2}\right)^{-6}.$$

Briefly comment on the stability of hydrogen burning in this white dwarf envelope.

4

(i) A close binary star has a circular orbit with angular velocity Ω and semi-major axis a. Star 2 behaves as a point mass while star 1 is extended, has a spin aligned with the orbital angular momentum and synchronized with the orbit. Star 1 is centrally condensed and of uniform composition. In a rotating frame in which the binary system is stationary let star 1 be at the origin and star 2 at $\boldsymbol{a}$. Show that surfaces of constant pressure P and density ρ are surfaces of constant potential

$$\Psi = -\frac{GM_1}{|\boldsymbol{r}|} - \frac{GM_2}{|\boldsymbol{r}-\boldsymbol{a}|} - \frac{GM}{2a^3}s^2,$$

where $M = M_1 + M_2$, $a = |\boldsymbol{a}|$ and s is the perpendicular distance to the rotation axis through the centre of mass of the system.

(ii) What is meant by the Roche lobe of star 1?

When star 1 is much smaller than its Roche lobe, interior equipotential surfaces S are approximately spherical and of radius r. Deduce that the mean effective gravitational acceleration within star 1 is

$$\langle g \rangle = \frac{1}{S}\int_S \boldsymbol{g}.\mathbf{d}\boldsymbol{S} \approx -\frac{GM_1}{r^2} + \frac{2}{3}\Omega^2 r,$$

where $\boldsymbol{g} = -\nabla\Psi$.

(iii) When the size of star 1 approaches that of its Roche lobe a better one-dimensional approximation for the equations of stellar structure can be constructed by considering Ψ as independent variable. Let $V(\Psi)$ be the volume enclosed by the equipotential surface $S(\Psi)$ and define an artificial radius r' such that $V(\Psi) = (4/3)\pi r'^3$. Show that we can write the mass equation as

$$\frac{\mathrm{d}m}{\mathrm{d}r'} = 4\pi r'^2 \rho$$

and hydrostatic equilibrium as

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -f_P \frac{Gm}{4\pi r'^4},$$

where

$$f_P = \frac{4\pi r'^4}{GmS\langle g^{-1}\rangle} \qquad \text{and} \qquad \langle g^{-1}\rangle = \frac{1}{S}\int_S g^{-1}\,\mathrm{d}S.$$

(iv) In radiative equilibrium the local flux

$$\boldsymbol{F} = -\frac{4acT^3}{3\kappa\rho}\nabla T,$$

where a is the radiation constant, c the speed of light, $T(\Psi)$ the temperature and κ the opacity. Show that the luminosity through an equipotential surface is

$$L_\Psi = -\frac{4acT^3}{3\kappa}S^2\langle g\rangle\langle g^{-1}\rangle\frac{\mathrm{d}T}{\mathrm{d}m}.$$

(v) Briefly describe why this formalism breaks down when star 1 overfills its Roche lobe.

END OF PAPER