UNIVERSITY OF CAMBRIDGE

MAMA/314, NST3AS/314, MAAS/314

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 9:00 am to 12:00 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

The paper will start with a reminder of key facts.
You may refer to these formulae in your solutions, but, please make sure to provide sufficient details when using them

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \qquad \nabla^2 \Phi = 4\pi G \rho. \tag{3}$$

Conservation laws for momentum and energy

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \hat{\Pi} = 0, \quad \hat{\Pi}_{ij} = \rho u_i u_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi}, \tag{4}$$

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{u^2}{2} + e \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{u} \left(\frac{u^2}{2} + h \right) + c \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0, \tag{5}$$

where h is the enthalpy obeying $\mathrm{d}h = T\ \mathrm{d}s + \rho^{-1}\mathrm{d}p$; $h = c_s^2/(\gamma - 1)$ for a polytropic gas with adiabatic index γ, where c_s is the speed of sound.

You may assume that for any scalar function f

$$\nabla f = \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z \quad \textit{(cylindrical coordinates)} \tag{6}$$

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \quad \textit{(spherical coordinates)}. \tag{7}$$

You may assume that for any vector $\mathbf{C}$ in cylindrical polar coordinates

$$\nabla \cdot \mathbf{C} = \frac{1}{R}\frac{\partial (R C_R)}{\partial R} + \frac{1}{R}\frac{\partial C_\phi}{\partial \phi} + \frac{\partial C_z}{\partial z}, \tag{8}$$

$$\nabla \times \mathbf{C} = \left(\frac{1}{R}\frac{\partial C_z}{\partial \phi} - \frac{\partial C_\phi}{\partial z} \right)\mathbf{e}_R + \left(\frac{\partial C_R}{\partial z} - \frac{\partial C_z}{\partial R} \right)\mathbf{e}_\phi + \frac{1}{R}\left(\frac{\partial (R C_\phi)}{\partial R} - \frac{\partial C_R}{\partial \phi} \right)\mathbf{e}_z, \tag{9}$$

and in spherical polar coordinates

$$\nabla \cdot \mathbf{C} = \frac{1}{r^2}\frac{\partial (r^2 C_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (C_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial \phi}, \tag{10}$$

$$\nabla \times \mathbf{C} = \frac{1}{r\sin\theta}\left(\frac{\partial (C_\phi \sin\theta)}{\partial \theta} - \frac{\partial C_\theta}{\partial \phi} \right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial C_r}{\partial \phi} - \frac{\partial (r C_\phi)}{\partial r} \right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial (r C_\theta)}{\partial r} - \frac{\partial C_r}{\partial \theta} \right)\mathbf{e}_\phi. \tag{11}$$

You may refer to these formulae in your solutions, but, please, make sure to provide sufficient details when using them.

1

(a) Define what it means for a magnetostatic structure to be in *force-free* equilibrium. Show that the magnetic field structure in a force-free equilibrium is governed by the equation

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \alpha \boldsymbol{B},$$

where α is a function of the spatial coordinates and c is the speed of light. Identify a constraint on $\alpha(\boldsymbol{r})$.

(b) Consider an axisymmetric magnetostatic configuration with properties depending on R only in cylindrical coordinates $\boldsymbol{r} = (R, \phi, z)$. Assume that the radial component B_R of $\boldsymbol{B}$ vanishes as $R \to 0$. Derive a closed form equation for the z-component of the magnetic field for general $\alpha(R)$.

(c) Now assume a particular form for $\alpha(R)$, namely

$$\alpha(R) = \frac{c}{4\pi} \frac{\kappa}{R},$$

where $\kappa > 0$ is a constant. Solve for the magnetic field (all components) not applying any boundary conditions. Show that for $\kappa > \kappa_c$, where κ_c is to be determined, the magnetic field exhibits an oscillatory radial dependence.

(d) Now consider the case of $\kappa = \kappa_c$. Find the general solution for B_z and demonstrate that the magnetic field lines make a constant angle with the z-axis as $R \to \infty$. Find this angle.

2

Consider the steady, hydrodynamic, transonic accretion of gas onto a point mass M. The accretion flow is spherically symmetric and obeys a globally isothermal equation of state $p = c_s^2\,\rho$, where p is pressure, ρ is density and c_s is the spatially constant sound speed. Far from the point mass (as $r \to \infty$) the gas is at rest and its density is ρ_0.

(a) Use the equations of motion and continuity to show that this accretion flow admits a sonic point at which the radial velocity of the flow u equals c_s. Find the radius of the sonic point r_s, assuming that the flow smoothly passes from subsonic to supersonic there.

(b) Using the equation of motion, or otherwise, demonstrate that

$$\mathcal{B} = \frac{u^2}{2} + c_s^2 \ln \rho + \Phi$$

is constant along the stream lines, where Φ is the gravitational potential of the point mass.

(c) Show that the mass accretion rate onto the point mass is

$$\dot{M} = A\rho_0 \frac{(GM)^2}{c_s^3}\,,$$

where A is the constant that you need to determine.

(d) Integrate the equations of motion and continuity, with the appropriate boundary conditions, to derive an algebraic, transcendental relation between $y = u/c_s$ and $x = r/r_s$ that does not contain any arbitrary constants or dimensional parameters.

(e) Using the relation obtained in part (d), or otherwise, determine the leading order expression for $u(r)$ assuming $r \ll r_s$ and the next order correction for its expansion in terms of $r/r_s \ll 1$.

3

A spacecraft in the shape of a thin, rectangular wing is moving through magnetised plasma. The wing lies in the $x-y$ plane of the Cartesian coordinate system (x, y, z), and its long side is parallel to the y-axis. The wing can be considered as infinitely thin in the z-direction and infinitely extended in the y-direction; it has a constant width L in the x-direction. The velocity of the wing is $\boldsymbol{u}_0 = u_0\mathbf{e}_x$. Due to its thinness, the wing experiences no aerodynamic drag as a result of its motion through the plasma.

Far ahead of the wing, plasma has an unperturbed density ρ_0 and is threaded by an unperturbed uniform magnetic field $\boldsymbol{B}_0 = B_0\mathbf{e}_z$. The gas has zero pressure, and there is no gravity.

The wing has a finite, constant surface conductivity Σ (i.e. the volumetric conductivity σ integrated over the small thickness of the wing in the z-direction), which is so low that the magnetic field penetrates into the wing with no significant distortion of its structure.

(a) Because of the non-zero conductivity of the wing, a current is driven through it. The surface current $\boldsymbol{J}$ (i.e. the current density $\boldsymbol{j}$ inside the wing integrated over its small thickness) obeys Ohm's law $\boldsymbol{J} = \Sigma\boldsymbol{E}$, where $\boldsymbol{E}$ is the electric field in the frame of the wing. Explain the origin of this current and determine the value and orientation of $\boldsymbol{J}$.

(b) Current running through the wing causes a perturbation in the surrounding plasma. Argue that this perturbation is stationary in the frame co-moving with the wing. Assuming the perturbations of the fluid variables are small compared to their background values, linearize the MHD equations and derive the following equation for the velocity perturbation δu_x in the x-direction

$$(u_A^2 - u_0^2)\frac{\partial^2 \delta u_x}{\partial x^2} + u_A^2\frac{\partial^2 \delta u_x}{\partial z^2} = 0, \tag{1}$$

where u_A is the unperturbed Alfven velocity.

(c) Use Maxwell's equation in the form $\nabla \times \boldsymbol{B} = (4\pi/c)\boldsymbol{j}$ to find the jump in δB_x caused by the current through the wing, where δB_x is the x-component of the magnetic field perturbation. Use this result to determine the boundary condition for δB_x at the upper and lower surfaces of the wing.

(d) Consider the limit of a super-Alfvenic velocity, $u_0 \gg u_A$. Use equation (1) and the boundary condition from part (c) to derive the solution for the perturbations of velocity and magnetic field in the plasma. Show that these perturbations can be non-zero only within a particular volume of space, the shape of which you must determine.

4

A supernova explosion drives a strong, spherically symmetric shock wave into the surrounding interstellar medium, which is at rest. The velocity of (shocked) gas inside the spherical shock behaves as a function of radius r and time t as

$$u(r,t) = u_0(t)\,\frac{r}{R(t)}, \qquad r < R(t),$$

where $R(t)$ is the radius of the shock and $u_0(t)$ is the post-shock gas velocity, i.e. $u_0(t) = u(R(t),t)$. The gas into which the shock propagates is cold (its thermal energy can be neglected) and has a constant density ρ_0; the adiabatic index of the gas is $\gamma = 5/3$.

(a) Use the velocity structure inside the shock to justify briefly whether or not you expect the evolution of the supernova remnant to be adiabatic.

(b) State (or derive) the post-shock values of the gas density $\rho(R(t),t)$ and velocity $u_0(t)$, assuming the shock to be strong and passage of the gas through the shock front to be adiabatic.

(c) Suppose now that the medium into which the shock propagates is pervaded by a uniform magnetic field $\boldsymbol{B}_0$, which is too weak to affect the fluid motion. Working in a spherical coordinate system (r,θ,ϕ) with the $\theta = 0$ axis aligned with $\boldsymbol{B}_0$, write down the post-shock magnetic field $\boldsymbol{B}(R(t),\theta,t)$ in spherical coordinates.

(d) Use the relevant MHD equations, the self-similarity of the $u(r,t)$ profile and the results of part (c) to determine the structure of the magnetic field $\boldsymbol{B}(r,\theta,t)$ inside the shock. Sketch the shape of the magnetic field lines inside and outside the shock.

END OF PAPER