UNIVERSITY OF CAMBRIDGE

MAMA/313, NST3AS/313, MAAS/313

MAT3
MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 9:00 am to 11:00 am

PAPER 313

SOLITONS, INSTANTONS AND GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that the nonlinear Schroedinger equation

$$i\phi_t = -\frac{1}{2}\phi_{xx} - |\phi|^2\phi + \epsilon V(\epsilon x)\phi \tag{1}$$

can be derived as the Euler-Lagrange equation for the action

$$S[\phi] = \int \Big[\langle i\phi, \dot{\phi}\rangle + \frac{1}{2}|\phi_x|^2 - \frac{1}{2}|\phi|^4 + \epsilon V(\epsilon x)|\phi|^2\Big]\,dxdt.$$

[*In this question* $\langle a, b\rangle = (\overline{a}b + a\overline{b})/2$.]

(i) *For the case* $\epsilon = 0$, show that if $E < 0$ then (1) has a solution $e^{-itE} f_E(x)$ with

$$f_E = \sqrt{2|E|}\,\mathrm{sech}\,(\sqrt{2|E|}x)\,.$$

(ii) Next, *still with* $\epsilon = 0$, show that this exact solution of (1) can be generalized to a solution $\phi(x;t) = \Phi(x; E, \theta(t), X(t), u(t))$ where

$$\Phi(x; E, \theta, X, u) = f_E(x - X)e^{i\theta + iu(x-X)} \tag{2}$$

with E constant, but θ, X, u evolving in time according to

$$\dot{\theta} = -E + \frac{1}{2}u^2\,, \quad \dot{X} = u\,, \quad \text{and} \quad \dot{u} = 0\,. \tag{3}$$

Calculate the energy and momentum at time t of these solutions, as defined, respectively, by the values of the functionals

$$H[\phi] = \int \Big[\frac{1}{2}|\phi_x|^2 - \frac{1}{2}|\phi|^4\Big]\,dx\,, \quad \text{(energy)} \qquad \text{and}$$
$$P[\phi] = \int \langle \phi, -i\phi_x\rangle\,dx\,, \quad \text{(momentum)}\,.$$

Calculate a relation between these two values, and comment briefly.

(iii) *For* $\epsilon = 1$ *and* $V(x) = \frac{1}{2}x^2$ find a generalization of (2)-(3) so that $\phi(x;t)$ as defined in (ii) solves (1).[*You should replace* f_E *in* (2) *by* $g_E : \mathbb{R} \to \mathbb{R}$ *which solves*

$$Eg_E = -\frac{1}{2}\frac{d^2 g_E}{dy^2} - g_E^3 + \frac{1}{2}y^2 g_E\,,$$

and modify (3) *as needed. You need* <u>*not*</u> *find* g_E *explicitly.*]

Comment briefly on your answer in relation to the classical Hamiltonian $H(x,p) = \frac{p^2}{2} + V(x)$.

(iv) *For small* ϵ, define and compute an effective Lagrangian to approximate the time evolution of the four parameters E, θ, X, u, and derive the predicted dynamics of the soliton in the potential $\epsilon V(\epsilon x)$.

2 Define the curvature $F \in \Omega^2(\mathbb{R}^4; su(2))$ associated to an $SU(2)$ connection $D = d + A$ on $\mathbb{R}^4$ determined by $A = \sum_{j=0}^3 A_j dx^j$, and derive the Bianchi identity which F satisfies. Show that solutions of the self-dual (sd) or anti-self-dual (asd) equations, $F = \pm * F$ give minimizers of

$$S_{\rm E} = -2 \int \text{Tr } (F \wedge *F)$$

for a given value of

$$k = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr } (F \wedge F)\ .$$

Find the minimum value of $S_{\rm E}$ in terms of k.

Now consider the case that the components of A are independent of x^0, and treat $\Phi = A_0$ as a new $su(2)$-valued scalar field; assume smoothness as required. Write down the Bianchi identity and the sd/asd equations in terms only of $\Phi, A_i, \partial_i \Phi$ and $B^i = \frac{1}{2}\epsilon^{ijk} F_{jk}$, where i, j, k run through $1, 2, 3$, and hence show that

$$\sum_{j=1}^3 D_j^2 \Phi = 0\,.$$

Hence, by computing $\Delta w = \sum_{j=1}^3 \partial_j^2 w$ where $w = 1 - |\Phi|^2$, deduce that solutions satisfying

$$|\Phi(\mathbf{x})| \to 1 \quad \text{as } |\mathbf{x}| \to +\infty$$

satisfy $|\Phi| \leqslant 1$ everywhere. Here $\mathbf{x} = (x^1, x^2, x^3)$.

[*In this question take* $\langle A, B \rangle = -2\text{Tr}\, AB$ *as inner product in* $su(2)$, *and as orthonormal basis* $T_a = \frac{i}{2}\sigma^a$ *satisfying* $[T_1, T_2] = -T_3$. *You may make use of the maximum principle: if a smooth* $w : \mathbb{R}^3 \to \mathbb{R}$ *satisfies* $\Delta w \leqslant 0$ *then* either w *is constant or if it attains a minimum value at a point* $x_0 \in \mathbb{R}^3$ *then* $w(x_0) > 0$.]

Also write down

$$-2\text{Tr } (F \wedge *F) \qquad \text{and} \qquad \text{Tr } (F \wedge F)$$

in terms of $\Phi, D_j\Phi$ and $B^i = \frac{1}{2}\epsilon^{ijk} F_{jk}$, and comment on the relation with

$$N = \lim_{R\to\infty} \frac{1}{4\pi} \oint_{|\mathbf{x}|=R} \langle \Phi\,, \mathbf{B}\cdot\mathbf{n} \rangle\, dS$$

(where $\mathbf{n}$ is the unit outward normal on the sphere $\{\mathbf{x} : |\mathbf{x}| = R\}$ and dS is the usual area measure).

Consider a solution of the form

$$\Phi^a(x) = f(r)\frac{x^a}{r}, \qquad A_i^a(x) = \epsilon_{iaj} x^j \alpha(r) \qquad r = |\mathbf{x}|\,. \tag{1}$$

Assuming that $f(0) = \alpha(0) = 0$ and

$$\lim_{r\to\infty} f(r) = 1 \qquad \text{and} \qquad \lim_{r\to\infty} r^2 \alpha(r) = +1\,, \tag{2}$$

calculate B^i and hence the number N.

3 The Abelian-Higgs energy functional on the disc $D = \{z : |z| < 1\} \subset \mathbb{C}$ is

$$V(A, \Phi) = \frac{1}{2} \int_D \Big[e^{-2\rho} (\partial_1 A_2 - \partial_2 A_1)^2 + |D\Phi|^2 + \frac{e^{2\rho}}{4} (1 - |\Phi|^2)^2 \Big] dx^1 dx^2 \,. \qquad (1)$$

Here $z = x^1 + ix^2$, ρ is a smooth real valued function on D, the metric on D is $g = e^{2\rho}\left((dx^1)^2 + (dx^2)^2\right)$ and the covariant derivative of $\Phi : D \to \mathbb{C}$ is $D\Phi = (\partial_j - iA_j)\Phi \, dx^j$. Carry out the Bogomolny argument to obtain the first order Bogomolny equations for the minimizers of (1).

With the area form $d\mu_g = e^{2\rho} dx^1 \wedge dx^2$ define and display explicitly the Hodge $*$-operator on p-forms for $p = 0, 1$ and 2. Find $B = *dA$ in terms of Φ for solutions of the Bogomolny equations.

Write down the form of the radial vortex solutions of winding number $N \in \mathbb{N}$ in polar coordinates (r, θ) with $z = re^{i\theta}$.

Now consider the case $e^{2\rho} = \frac{8}{(1-|z|^2)^2}$, and define ψ by

$$\psi = u - \ln(1 - z\bar{z}) + \ln 2$$

where $u = \ln |\Phi|$. Show that away from the zeros of Φ there holds $\Delta\psi = e^{2\psi}$. Verify that this latter equation has solution

$$\psi = \frac{1}{2} \ln \frac{4|g'(z)|^2}{(1 - |g(z)|^2)^2}$$

for any holomorphic function $g : D \to D$, and hence obtain an explicit radially symmetric vortex solution for each positive integer value N of the winding number. Calculate directly the magnetic flux $\int B d\mu_g$ of your solution in case $N = 1$, and verify the relation $\int B d\mu_g = 2\pi$ in this case.

END OF PAPER