UNIVERSITY OF CAMBRIDGE

MAMA/312, NST3AS/312, MAAS/312

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 9:00 am to 12:00 pm

PAPER 312

FIELD THEORY IN COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In de Sitter spacetime, consider the action

$$S = -\int d^3x d\eta \frac{a^4}{2}\left[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right] ,$$

where $a(\eta)$ is the scale factor and $m^2 = 2H^2$ with H the constant Hubble parameter.

(i) Compute the equation of motion for $\phi(\eta, \mathbf{k})$ and find the most general solution.

(ii) Using that ϕ has mode functions

$$f(k,\eta) = \frac{H}{\sqrt{2k}}\eta e^{-ik\eta} ,$$

write down expressions for the bulk-boundary and bulk-bulk propagators

$$\begin{aligned} \bullet\!- = G_r(\eta, p) &= \langle 0|\,\phi(\eta_0, \mathbf{p})\phi(\eta, \mathbf{p}')\,|0\rangle' \\ \bullet - \bullet = G_{rr}(\eta_1, \eta_2, p) &= \langle 0|\,T\phi(\eta_1, \mathbf{p})\phi(\eta_2, \mathbf{p}')\,|0\rangle' \\ \circ - \bullet = G_{lr}(\eta_1, \eta_2, p) &= \langle 0|\,\phi(\eta_1, \mathbf{p})\phi(\eta_2, \mathbf{p}')\,|0\rangle' \end{aligned}$$

where a prime on a correlator means that we dropped the factor $(2\pi)^3\delta(\mathbf{p}+\mathbf{p}')$.

(iii) Now consider adding to the action the quartic interaction

$$S_{int} = \int d^3x d\eta\, a^4\, \frac{\lambda}{4!}\phi^4 .$$

For the six-point correlator B_6, draw the Feynman diagram corresponding to the exchange contribution at order λ^2 from fields with external momenta $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ to $\mathbf{k}_4$, $\mathbf{k}_5$ and $\mathbf{k}_6$. You should use the notation

$$k_L = k_1 + k_2 + k_3 , \qquad k_R = k_4 + k_5 + k_6 , \qquad k_T = \sum_{a=1}^{6} k_a ,$$

$$k_I = |\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3| , \qquad E_L = k_L + k_I , \qquad E_R = k_R + k_I .$$

How are the contributions $B_6^{(rr)}$ and $B_6^{(lr)}$ related to the contributions $B_6^{(ll)}$ and $B_6^{(rl)}$? From the Feynman rules compute $B_6^{(rr)}$ and $B_6^{(lr)}$ and hence B_6 as a function of the six external momenta $\mathbf{k}_a$ for $a = 1, \ldots, 6$. Show that if the correlator is evaluated at late time with $\eta_0 \to 0$ then, to leading order, your results take the form

$$B_6^{(rr)} = A\frac{\eta_0^6}{k_T}\left(\frac{1}{E_R} + \frac{1}{E_L}\right) , \qquad B_6^{(lr)} = B\frac{\eta_0^6}{E_R E_L} ,$$

where A and B are quantities you should determine.

[*Hint: You may take $\eta_0 = 0$ in all boundaries of integration.*]

2

Consider a free massless scalar ϕ in de Sitter spacetime,

$$S = -\int d^3x d\eta \, \frac{a^4}{2} \, \partial_\mu \phi \partial^\mu \phi \, .$$

(i) Derive the equations of motion for ϕ. Show that homogeneous solutions $\phi(\mathbf{x}, \eta) = \bar{\phi}(\eta)$ obey $\partial_\eta \bar{\phi} \propto a^p$ where p is a constant you should determine.

(ii) Using the background Friedmann equations,

$$3H^2 M_{\rm Pl}^2 = \frac{1}{2}\dot{\phi}^2 + \Lambda \, , \qquad\qquad -\dot{H} M_{\rm Pl}^2 = \frac{1}{2}\dot{\phi}^2 \, ,$$

where Λ is a constant, compute the slow-roll parameters $\epsilon = -\dot{H}/H^2$ and $\eta_{\rm SR} = \dot{\epsilon}/(\epsilon H)$ (not to be confused with conformal time η). Using the result of part (i) show that at late times in an expanding universe, i.e. for $a \to \infty$, one finds the leading behaviour $aH \simeq -1/\eta$, $\epsilon \simeq 0$ and $\eta_{\rm SR} \simeq -6$. This solution is known as "ultra slow-roll" inflation.

(iii) The quadratic action for $\mathcal{R}$ is

$$S = \int d^3x d\eta \, \frac{M_{\rm Pl}^2}{2} \, z(\eta)^2 \, \left[(\mathcal{R}')^2 - \partial_i \mathcal{R} \delta_{ij} \partial_j \mathcal{R} \right] \, ,$$

where $z(\eta) = a\sqrt{2\epsilon}$. Derive the equation of motion for $\mathcal{R}$. Using previous results compute the asymptotic value of $\partial_\eta z / z$ for $a \to \infty$. A solution for $\mathcal{R}$ at late times takes the form

$$A \frac{(1 + ik\eta)}{\eta^q} e^{-ik\eta} \, ,$$

where A is time-independent and q is a real constant. Considering the limit $|\mathbf{k}| \to 0$ of the equation of motion or otherwise, determine q for the leading solution.

(iv) After quantizing $\mathcal{R}$, compute the following two-point correlators for any η

$$\langle \mathcal{R}(\mathbf{k}, \eta) \mathcal{R}(\mathbf{k}', \eta) \rangle \, , \langle \mathcal{R}(\mathbf{k}, \eta) \dot{\mathcal{R}}(\mathbf{k}', \eta) \rangle \, , \langle \dot{\mathcal{R}}(\mathbf{k}, \eta) \dot{\mathcal{R}}(\mathbf{k}', \eta) \rangle \, .$$

Hence extract the behaviour at late times of the classicality parameter

$$C(k, \eta) = \frac{|\langle [\mathcal{R}(\mathbf{k}, \eta), \dot{\mathcal{R}}(\mathbf{k}', \eta)] \rangle'|}{\sqrt{\langle \mathcal{R}(\mathbf{k}, \eta) \mathcal{R}(\mathbf{k}', \eta) \rangle' \langle \dot{\mathcal{R}}(\mathbf{k}, \eta) \dot{\mathcal{R}}(\mathbf{k}', \eta) \rangle'}} \, ,$$

where here a prime on a correlator means that we dropped the factor $(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')$. Comment on this result and contrast it with the case of slow-roll inflation.

3

In standard perturbation theory, the dynamics of dark matter is described by

$$\delta' + \nabla \cdot [(1+\delta)\,\mathbf{v}] = 0\,, \qquad v_i' + \mathcal{H} v_i + (\mathbf{v}\cdot\nabla)\,v_i = -\nabla_i \phi\,, \qquad \nabla^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_m \delta\,.$$

(i) Assuming zero vorticity, derive the $\alpha(\mathbf{q}_1, \mathbf{q}_2)$ and $\beta(\mathbf{q}_1, \mathbf{q}_2)$ kernels characterizing the non-linearities of the continuity and Euler equations, respectively.

(ii) The SPT kernel $F_2(\mathbf{q}_1, \mathbf{q}_2)$ takes the following form

$$F_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{5}{7} + A\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1 k_2}\left(\frac{k_2}{k_1} + \frac{k_1}{k_2}\right) + \frac{2}{7}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2 k_2^2}\,,$$

where A is a numerical constant. State the UV behavior of $F_2(\mathbf{q}, \mathbf{k}-\mathbf{q})$ for fixed $\mathbf{k}$ and $|\mathbf{q}| \to \infty$. Hence determine the constant A.

(iii) Draw all the standard perturbation theory diagrams contributing to the connected 5-point function of matter density perturbations at tree level. For each diagram, you need to consider only a single labelling of external momenta, rather than specifying all possible permutations. Show that all these diagrams scale as P_{lin}^n, where P_{lin} is the linear power spectrum, and determine the integer n. Write down an algebraic expression for the diagram containing only the SPT kernels F_1 and F_2.

4

The linear-order Boltzmann equation for photons is

$$\Theta' + \hat{\mathbf{p}} \cdot \nabla\Theta = \Phi' - \hat{\mathbf{p}} \cdot \nabla\Psi - \Gamma\left(\Theta - \Theta_0 - \hat{\mathbf{p}} \cdot \mathbf{v}_e\right) .$$

(i) Show how to derive from this expression the line-of-sight solution

$$e^{-\tau}(\Theta + \Psi)\Big|_0^{\eta_0} = \int_0^{\eta_0} d\eta' \; \hat{S}(\eta', \mathbf{x}_0 + (\eta_0 - \eta')\hat{\mathbf{n}}, \hat{\mathbf{n}}) \, ,$$

where $\hat{S}$ is a source you should specify and η_0 and $\mathbf{x}_0$ are the time and position of observation.

(ii) Recall the definition of the visibility function and of the optical depth

$$g(\eta) = \partial_\eta e^{-\tau(\eta)} \, , \qquad\qquad \tau = \int_\eta^{\eta_0} d\eta' \; \Gamma(\eta') \, .$$

Now assume a hypothetical universe in which a fraction $0 \leqslant p \leqslant 1$ of CMB photons we see from earth today at η_0 last scattered at recombination $\eta_\star$, and the remaining fraction last scattered at η_1 with $\eta_\star < \eta_1 < \eta_0$. Write down the corresponding visibility function and the quantity $e^{-\tau(\eta)}$. Plot $e^{-\tau(\eta)}$ as a function of time and describe its physical meaning.

(iii) Under the assumption of part (ii), evaluate the line-of-sight integral. Briefly describe in words what equations you would need to solve to evaluate the result in terms of primordial initial conditions.

END OF PAPER