MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 9:00 am to 12:00 pm

PAPER 311

BLACK HOLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define what it means for a spacetime to be *static* and explain how to introduce coordinates in such a spacetime so that the metric takes the form $ds^2 = g_{00}(x)dt^2 + g_{ij}(x)dx^i dx^j$ with $g_{00} < 0$.

(b) Explain carefully how to construct the Kruskal extension of the Schwarzschild spacetime. Let k be the Killing vector field associated with Schwarzschild time translations. What is k in Kruskal coordinates?

(c) On a large Kruskal diagram, show the following (with clear labelling):

(i) an ingoing radial null geodesic; (ii) an outgoing radial null geodesic; (iii) an extendible causal curve; (iv) an incomplete, inextendible, causal geodesic; (v) a complete, inextendible, causal geodesic.

[*You may add a few words of explanation if necessary.*]

(d) Alice, Bob and Carol are astronauts on a rocket at a fixed position with radius $R > 2M$ outside a Schwarzschild black hole. At a certain instant, Bob sets his watch to read 0 and Alice immediately leaves the rocket and falls radially into the black hole. Later, when Bob's watch reads time T, he also leaves the rocket and falls radially into the black hole, along an identical trajectory. Carol, who is more sensible, remains in the rocket.

(i) As he crosses the event horizon, Bob looks at Alice. By how much is she redshifted?

[*Hints. Argue that Bob's trajectory is related to Alice's trajectory by an isometry. Use this to relate the Kruskal coordinate V_A, V_B at which each crosses the horizon. Recall that an observer with velocity U^a measures the energy of a photon with momentum P^a to be $-U \cdot P$.*]

(ii) Carol observes Alice during her journey. Describe in two sentences what she sees.

2

(a) Let Σ be a surface of constant r *inside* a Schwarzschild black hole (so $r < 2M$).

(i) Show that this surface, with its induced metric, is isometric to a cylinder $\mathbb{R} \times S^2$ in four-dimensional Euclidean space.

(ii) Compute the extrinsic curvature of Σ, expressing your answer in Schwarzschild coordinates. [*You may wish to use* $K_{ab} = h_a^c h_b^d \nabla_c n_d$.]

(iii) View Σ with its induced metric and extrinsic curvature as initial data for the vacuum Einstein equation. What is the maximal Cauchy development of this initial data? Is it extendible? Discuss, in a few sentences, whether this is consistent with the strong cosmic censorship conjecture.

(b) Explain how to construct the Penrose diagram of four-dimensional Minkowski spacetime.

(c) The conformal wave equation is $\nabla^a \nabla_a \psi - (1/6)R\psi = 0$. For the rest of this question you may assume that if ψ is a solution of this equation in a spacetime with metric g then $\bar{\psi} \equiv \Omega^{-1}\psi$ is a solution of the same equation in a spacetime with metric $\bar{g} = \Omega^2 g$. We will take g to be the Minkowski metric (with $R = 0$ so ψ satisfies the standard wave equation) and $\bar{g}$ to be the metric of the Einstein static universe.

Assume that $\bar{\psi}$ is globally defined on the Einstein static universe.

(i) Show that $v\psi$ has a well-defined limit at $\mathcal{I}^+$ and $u\psi$ has a well-defined limit at $\mathcal{I}^-$, where u and v are the usual retarded and advanced time coordinates.

(ii) What is the corresponding result at $i^{\pm}$? (Your answer should be non-zero for generic $\bar{\psi}$.)

3

(a) An isolated star undergoes gravitational collapse, during which a trapped surface forms. Explain why it is believed that a Kerr black hole will form. You should describe briefly suitable theorems and conjectures in your answer.

(b) Let $\mathcal{N}$ be a null hypersurface with normal n_a.

(i) Prove that n^a is tangent to null geodesics (the generators of $\mathcal{N}$).

(ii) Consider a null geodesic congruence containing the generators of $\mathcal{N}$. Prove that the rotation of this congruence vanishes on $\mathcal{N}$.

(c) The following generalization of the Kerr metric is a charged solution of the equations of motion of low energy string theory:

$$g = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+2br+a^2)d\phi - adt\right]^2 + \Sigma d\theta^2$$

where

$$\Delta = r^2 - 2(m-b)r + a^2 \qquad \Sigma = r^2 + 2br + a^2\cos^2\theta$$

and m, a, b are constants. Assume that Δ has real roots $r_\pm$ with $r_+ > r_-$ and $r_+ > 0$.

(i) Define new coordinates as follows:

$$dv = dt + A(r)dr \qquad d\chi = d\phi + B(r)dr$$

where the functions A, B both have a simple pole at $r = r_+$. Show that A, B can be chosen such that the metric in coordinates (v, r, θ, χ) can be analytically continued through $r = r_+$.

[*Hint: does smoothness of* $g_{r\chi}$ *suggest a relation between* A *and* B*?*]

(ii) Prove that the surface $r = r_+$ is a Killing horizon and calculate its angular velocity.

[*Hint: look for a linear combination of Killing fields* ξ^a *such that* $\xi_\mu|_{r=r_+}$ *has only a* r*-component.*]

(iii) Calculate the surface gravity of this Killing horizon for $a = 0$.

4

(a) Prove the version of the first law of black hole mechanics that relates the mass and angular momentum of infalling matter to the change in the area of the event horizon. (You may assume Raychaudhuri's equation, properties of Gaussian null coordinates and properties of the surface gravity.)

(b) In the Penrose process, a particle of energy E and angular momentum L falls into a Kerr black hole. Use the first and second laws of black hole mechanics to show that $\Omega_H L \leqslant E$. Explain why a particle with negative E cannot escape from the ergosphere.

(c) Explain why the discovery that black holes emit thermal radiation implies that the laws of black hole mechanics can be reinterpreted in thermodynamical terms. A radiating black hole shrinks, in violation of the second law of black hole mechanics. What assumption in the statement of the second law is violated by this process? Why does this shrinking not imply a violation of the second law of thermodynamics?

END OF PAPER