MAMA/310, NST3AS/310, MAAS/310

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2024 9:00 am to 12:00 pm

PAPER 310

COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) From the continuity equation for the energy density, determine the scaling of the energy density ρ with scale factor a for a component with constant equation of state parameter w. Hence show that the Hubble parameter can be written as

$$H(z) = H_0 \left[\sum_i \Omega_{i,0} (1+z)^{3(1+w_i)} \right]^{1/2} , \tag{1}$$

where the sum is over components i with constant equation of state parameters w_i and where you should define $\Omega_{i,0}$.

(b) Now consider a universe containing only matter (m) and a non-standard dark energy component (DE); the non-standard dark energy component has an equation of state parameter $w(z)$ that is not constant, but instead depends on redshift. Show that in such a universe the Hubble parameter is given by

$$H(z) = H_0 \left[\Omega_{m,0}(1+z)^3 + \Omega_{DE,0}(1+z)^3 \exp\left(\int_0^z \frac{w(z')}{X(z')} dz' \right) \right]^{1/2} , \tag{2}$$

where $X(z)$ is a function you should specify.

(c) The flux F received from a supernova with a luminosity L is given by $F = \frac{L}{(1+z)^2 4\pi \chi^2(z)}$, where $\chi(z)$ is the comoving distance to the supernova (which has a redshift z). You may assume that the supernova luminosity takes the same constant value for all supernovae. Hence explain carefully how measurements of supernovae can be used to constrain the equation of state of dark energy $w(z)$. Does the value of L need to be known for $w(z)$ to be constrained?

(d) A certain population of galaxies forms at a known, fixed cosmic time. Assume that from our knowledge of galaxy evolution as well as spectroscopic information, we can determine the age of each of the observed galaxies when their light was emitted. Derive a relation between the difference in redshift Δz and difference in age Δt of a pair of galaxies and the Hubble parameter $H(z)$ (you may assume that $\Delta z \ll 1$), and hence explain how measurements of galaxy ages at different redshifts can be used to constrain the equation of state of dark energy $w(z)$. Why could this probe be more sensitive to rapid, oscillatory variations in $w(z)$ with redshift than supernova measurements?

2

(a) Explain in detail why the temperature T_ν of the cosmic neutrino background is related to the CMB photon temperature T today by

$$\frac{T_\nu}{T} = \left(\frac{4}{11}\right)^{1/3}. \tag{1}$$

In your explanation, you may assume without proof that the entropy $S = \frac{\rho+P}{T}V$ is conserved in an expanding universe.

(b) Deduce that the energy density in radiation is given by

$$\rho_r = \frac{\pi^2}{30}\left[2 + \frac{7}{8} \times 2 \times \left(\frac{4}{11}\right)^{4/3} \times N_{eff}\right] T^4, \tag{2}$$

where N_{eff} is the (effective) number of neutrino species. In deriving this expression, you may assume relativistic standard model neutrinos.
[Hint: you may assume the standard expression for the energy density of a relativistic particle, $\rho = \frac{\pi^2}{30} gCT^4$, where g is the number of spin degrees of freedom and where $C = 1$ for bosons and $C = 7/8$ for fermions. You may also neglect any influence of electron positron annihilation on the decoupled neutrinos.]

(c) Now consider a relativistic scalar particle that, while initially in equilibrium with the photons and other coupled particles, decouples at a higher temperature T_d than a standard model neutrino. Derive an expression for the additional contribution to N_{eff} produced by this particle as a function of the effective number of entropy degrees of freedom for all other particles $g_{*S}(T_d)$.

(d) A model of new physics is proposed that involves a very large number of additional particle species. At very high temperatures > 1000 GeV, these particles are all relativistic and in equilibrium with the standard model. As the temperature begins to fall, only one of the many new particle species, a light scalar particle, is predicted to decouple from the thermal bath. At temperatures ≈ 200 GeV and below only the standard model particles and the new light scalar are present. A measurement of N_{eff} with a precision of 0.1% is made, and no evidence for departures from the standard model is found at this level of precision. Can we definitively exclude the proposed new physics model based on this measurement?
[Hint: you may assume that the maximum g_{*S} reached at high temperatures (at and above ≈ 200 GeV) in the standard model is $g_{*S} = 106.75$.]

3 This question is concerned with the effect of dark energy on the growth of cosmic structure on subhorizon scales. You may assume standard dark energy with an equation of state parameter $w = -1$ throughout the question.

(a) The perturbed energy momentum conservation and Poisson equations on subhorizon scales are given by:

$$\delta_m' + 3\mathcal{H}\left(\frac{\delta P_m}{\delta\rho_m} - \frac{\bar{P}_m}{\bar{\rho}_m}\right)\delta_m = -\left(1 + \frac{\bar{P}_m}{\bar{\rho}_m}\right)(\nabla\cdot\mathbf{v} - 3\Phi'), \tag{1}$$

$$\mathbf{v}' + 3\mathcal{H}\left(\frac{1}{3} - \frac{\bar{P}_m}{\bar{\rho}_m}\right)\mathbf{v} = -\frac{\nabla\delta P_m}{\bar{\rho}_m + \bar{P}_m} - \nabla\Phi, \tag{2}$$

$$\nabla^2\Phi = 4\pi G a^2 \sum_i \bar{\rho}_i \delta_i. \tag{3}$$

Here primes indicate derivatives with respect to conformal time, superscript bars indicate unperturbed background quantities, the index i labels the components of the universe, subscript m labels the matter component, $\mathcal{H}$ is the conformal Hubble parameter, δ_i is the fractional density contrast for component i, Φ gives the standard Newtonian gauge potential perturbation, $\mathbf{v}$ is the matter velocity field, and P and ρ indicate pressure and energy density, respectively.

From these equations, derive the following expression for the evolution of the matter fractional density contrast δ_m during matter domination:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0, \tag{4}$$

where superscript dots indicate derivatives with respect to time. You may assume without proof that this result is also valid during dark energy domination.

(b) Hence derive how δ_m evolves with the scale factor a during matter domination and during dark energy domination.

(c) Now suppose that both dark energy and dark matter are present and not necessarily negligible (you may assume that radiation can be neglected). Show that in this scenario we obtain the following evolution equation written in terms of derivatives with respect to the scale factor a:

$$\frac{d^2\delta_m}{da^2} + \left(\frac{d\ln H}{da} + \frac{3}{a}\right)\frac{d\delta_m}{da} - \frac{3\Omega_{m,0}H_0^2}{2a^5H^2}\delta_m = 0. \tag{5}$$

(d) By substituting the variable $u = \delta_m/H$ or otherwise, deduce an expression for the growth of δ_m with scale factor (you may leave your expression in terms of an integral). *[Hint: after substitution, you may wish to show that the term proportional to u vanishes.]*

(e) Assume that galaxies form from perturbations of a certain characteristic scale when these density perturbations become of order unity in size (and hence non-linear collapse takes place.) Assume also that such density perturbations are measured to have an amplitude $\delta_m = A$ at the time of CMB decoupling, where A is a small constant satisfying $A < 1$. Under these assumptions, can the existence of galaxies be used to place an upper limit on the dark energy density ρ_Λ? Justify your answer.

4 Consider a standard single-field slow-roll inflation model, where ϕ is the inflation field and $V(\phi)$ is its potential. You may assume throughout this question that $a(\tau) = -(H\tau)^{-1}$ (with τ the conformal time) and that $H = \sqrt{\frac{V(\phi)}{3M_{\rm pl}^2}} \approx$ constant.

(a) Canonical quantization leads to the following expression for the field operator $\hat{f} = a\hat{\delta\phi}$, describing perturbations to the inflation field $\delta\phi$:

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[f_{\mathbf{k}}(\tau) \hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + f^*_{\mathbf{k}}(\tau) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right]$$

where $f^*_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau})$ and $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}$ are lowering and raising operators. State the commutation relations obeyed by $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}'}$. By calculating the two point correlation function of $\delta\phi$, deduce the dimensionless power spectrum of $\delta\phi$. Evaluate it when $k \ll aH$, and show that the spectrum is given by

$$\Delta^2_{\delta\phi} = \left(\frac{H}{2\pi} \right)^2 . \tag{1}$$

[*Hint: you may assume that the dimensionless power spectrum $\Delta^2_{\delta\phi}$ is related to the two point correlation function via $\langle 0|\hat{\delta\phi}(\tau, \mathbf{x})\hat{\delta\phi}(\tau, \mathbf{x} + \mathbf{r})|0\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{2\pi^2}{k^3} \Delta^2_{\delta\phi} e^{-i\mathbf{k}\cdot\mathbf{r}}$*]

(b) The power spectrum of the comoving curvature perturbation $\mathcal{R}$ in this inflation model is hence given by

$$\Delta^2_{\mathcal{R}}(k) = \frac{1}{2\epsilon \mathrm{M}^2_{\rm pl}} \left(\frac{H}{2\pi} \right)^2 , \tag{2}$$

where $\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{(\dot{\phi})^2}{2H^2 \mathrm{M}^2_{\rm pl}}$ is the Hubble slow-roll parameter. Specify when the right hand side of this equation is to be evaluated; then show that the scalar spectral index $n_s \equiv 1 + \frac{d \ln \Delta^2_{\mathcal{R}}}{d \ln k}$ is given by

$$n_s - 1 = -2\epsilon - \eta, \tag{3}$$

where $\eta = \frac{d \ln \epsilon}{dN}$ is the second Hubble slow roll parameter.

(c) The predicted level of tensor perturbations can also be computed in a manner analogous to the scalar perturbation case; the result is that the power spectrum of tensor perturbations is given by $\Delta^2_t(k) = \frac{8}{\mathrm{M}^2_{\rm pl}} \left(\frac{H}{2\pi} \right)^2$ (with the right hand side evaluated at the same time as in the scalar result). Show that the tensor spectral index $n_T \equiv \frac{d \ln \Delta^2_t}{d \ln k}$ is given by $n_T = -C\epsilon$, where C is a positive constant you should specify. Hence derive a relation between the tensor-to-scalar ratio $r \equiv \Delta^2_t(k)/\Delta^2_{\mathcal{R}}(k)$ and n_T.

(d) Motivated by the fact that tensor modes have not yet been detected in the CMB on large scales, some researchers have considered whether tensor modes could have a power spectrum with $n_T > 0$, referred to as a "blue" spectrum. Discuss whether a blue spectrum of tensor modes can arise from a simple, standard inflation model in which a single field ϕ rolls slowly down a potential $V(\phi)$. Discuss also whether such a simple inflation model can produce a blue spectrum of scalar perturbations ($n_s - 1 > 0$).

END OF PAPER