UNIVERSITY OF CAMBRIDGE

MAMA/309, NST3AS/309, MAAS/309

MAT3

MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

Throughout, units are chosen such that $c = G = 1$, and the signature convention for spacetime is $(-+++)$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Let (M, g) be a Lorentzian spacetime, and $\gamma : [0,1] \to M$ a smooth timelike curve with $\gamma(0) = p$, $\gamma(1) = q$. Assume we work in a local coordinate system $\{x^\mu\}$ such that γ is mapped to the curve $x^\mu = x^\mu(s)$ for $0 \leqslant s \leqslant 1$.

i) Write down an expression for the proper time T elapsed along the curve from p to q.

ii) Show that if γ extremizes T among timelike curves with the same endpoints, then in local coordinates it must satisfy the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_\nu{}^\mu{}_\sigma \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0,$$

where τ is the proper time along the curve and you should give an expression for $\Gamma_\nu{}^\mu{}_\sigma$ in terms of $g_{\mu\nu}$.

iii) Show that the geodesic equation may also be obtained as the Euler-Lagrange equation of the action

$$S[x] = \frac{1}{2}\int_0^T g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau.$$

b) Now consider the gravitational plane wave solution to the vacuum Einstein equations:

$$g = -dt^2 + dx^2 + dy^2 + dz^2 + 2xyA(t-z)(dt-dz)^2$$

where $A(u)$ is a smooth function satisfying $A(u) = 0$ for $|u| > U$ for some $U > 0$.

i) Sketch in the $(t, z)-$plane the regions where g is locally isometric to Minkowski spacetime. Explain briefly why this spacetime represents a pulse of gravitational radiation of length $2U$ moving in the $z-$direction.

ii) Using the result of *a)iii)* above, or otherwise, write down the equations satisfied by a geodesic of this metric, and show that $\dot{t}(\tau) - \dot{z}(\tau)$ is constant along any geodesic.

iii) A test mass falls along the timelike geodesic given for $\tau < -U$ by

$$(t(\tau), x(\tau), y(\tau), z(\tau)) = (\tau, x_0, y_0, 0)$$

Assume that $A(u) = \epsilon a(u)$ where $0 < \epsilon \ll 1$. Find, to first order in ϵ, $x(\tau)$ and $y(\tau)$ for all τ and show that for $\tau > U$

$$x(\tau) = x_0 + \delta x + \tau\delta v_x$$

where

$$\delta x = -\epsilon y_0 \int_{-\infty}^{\infty} ua(u)du, \qquad \delta v_x = \epsilon y_0 \int_{-\infty}^{\infty} a(u)du$$

and give a similar expression for $y(\tau)$. Show that $z(\tau)$ vanishes to first order in ϵ.

[Hint: You may wish to use the identity $\int_{-\infty}^{\tau} f(u)du = \tau f(\tau) - \int_{-\infty}^{\tau} uf'(u)du$, valid for f decaying rapidly as $x \to -\infty$]

iv) Suppose a satisfies $\int_{-\infty}^{\infty} a(u)du = 0$, $\int_{-\infty}^{\infty} ua(u)du > 0$. Sketch the position of a set of test masses initially arranged at rest in a circle around the origin in the plane $\{z = 0\}$ before and after the passage of the pulse.

2 Suppose that (M, g) is a smooth Lorentzian manifold in $(1+3)$-dimensions. Let ∇ be the Levi-Civita connection.

a) Suppose V^a is a smooth vector field on M and ω_a a smooth covector field. Starting from the fact that $(\mathcal{L}_X Y)^a = [X, Y]^a$ for any vector fields X, Y, and assuming the Leibniz rule for the Lie derivative, show

$$(\mathcal{L}_V \omega)_a = V^b \nabla_b \omega_a + \omega_b \nabla_a V^b$$
$$(\mathcal{L}_V g)_{ab} = \nabla_a V_b + \nabla_b V_a.$$

[Hint: You may assume the existence of normal coordinates at a point p.]

b) Suppose $F_{ab} = -F_{ba}$ solves the vacuum Maxwell equations:

$$\nabla_a F^a{}_b = 0, \qquad \nabla_{[a} F_{bc]} = 0,$$

and let $T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}$ be the corresponding energy-momentum tensor.

i) Show that T_{ab} is conserved and traceless, i.e. $\nabla_a T^a{}_b = 0$ and $T^a{}_a = 0$. Deduce that for any smooth vector field V^a, $\nabla_a (T^{ab} V_b) = \frac{1}{2} T^{ab} (\mathcal{L}_V g)_{ab}$.

ii) Show that if V^a is a future directed timelike vector field, and W^a is a future directed timelike or null vector field then $T_{ab} V^a W^b \geqslant 0$.
[Hint: At a point p, you may assume without loss of generality that $g_{\mu\nu} = \eta_{\mu\nu}$, $V^\mu = (1, 0, 0, 0)$ and $W^\nu = (1, w, 0, 0)$ where $|w| \leqslant 1$.]

c) Let $M = \mathbb{R}_u \times (0, \infty)_r \times (0, \pi)_\theta \times (0, 2\pi)_\phi$ and assume g is the Minkowski metric in outgoing null coordinates:

$$g = -du^2 - 2du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where we take $\partial/\partial u$ to be future directed. Let $\Sigma_t = \{(u, r, \theta, \phi) \in M | u + r = t\}$.

i) Find n^a, the unit future directed normal of Σ_t in the (u, r, θ, ϕ) coordinate basis, together with the volume form $d\sigma$ induced by g on the surface.

ii) Let V^a be the vector field given in these coordinates by $r\frac{\partial}{\partial r}$. Show that

$$\mathcal{L}_V dr = dr, \quad \mathcal{L}_V du = \mathcal{L}_V d\theta = \mathcal{L}_V d\phi = 0,$$

and deduce that
$$(\mathcal{L}_V g)_{ab} = 2 g_{ab} + \frac{\alpha}{r} n_{(a} V_{b)}$$

for some constant α which you should determine. Deduce that $\nabla_a (T^{ab} V_b) \geqslant 0$.

iii) By integrating $\nabla_a (T^{ab} V_b)$ over the region $S = \{(u, r, \theta, \phi) : t_0 \leqslant u + r \leqslant t_1\}$ and assuming the fields decay sufficiently rapidly as $r \to \infty$, show that

$$\varepsilon(t) = \int_{\Sigma_t} n_a T^{ab} V_b \, d\sigma$$

is a non-negative, monotone decreasing, function of t.

3

a) i) For a variation of the metric $g_{ab} \to g_{ab}+\delta g_{ab}$ derive formulae for the variations of the volume form, the inverse metric and the Christoffel symbols.

ii) Show that $\delta R = -R^{ab}\delta g_{ab} + \nabla_a X^a$, where $X^a = g^{cb}\delta\Gamma_c{}^a{}_b - g^{ab}\delta\Gamma_b{}^c{}_c$. Deduce that

$$\delta R = -R^{ab}\delta g_{ab} + \alpha g^{ab}\nabla_c\nabla^c\delta g_{ab} + \beta\nabla^a\nabla^b\delta g_{ab}.$$

where α, β are constants to be determined.

[In a coordinate basis $R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma_\nu{}^\mu{}_\sigma - \partial_\sigma\Gamma_\nu{}^\mu{}_\rho + \Gamma_\nu{}^\tau{}_\sigma\Gamma_\tau{}^\mu{}_\rho - \Gamma_\nu{}^\tau{}_\rho\Gamma_\tau{}^\mu{}_\sigma$*]*

b) A proposed modification of Einstein's theory is derived from the action

$$S = \frac{1}{16\pi}\int_M f(R)d\mathrm{vol}_g + S_{matter}.$$

Here f is a given smooth function with $f(0) = 0$, and S_{matter} is an action for the matter fields. Assume that under a variation of the metric $S_{matter} \to S_{matter} + \frac{1}{2}\int_M T^{ab}\delta g_{ab} d\mathrm{vol}_g$.

i) Show that requiring δS to vanish under any variation of the metric vanishing outside a bounded region gives the field equations

$$f'(R)R_{ab} - \frac{1}{2}g_{ab}f(R) + [\alpha' g_{ab}\nabla_c\nabla^c + \beta'\nabla_a\nabla_b]f'(R) = 8\pi T_{ab}$$

for some constants α', β' which you should again determine.

ii) Show that in the absence of matter, the Minkowski spacetime satisfies the field equations.

iii) Suppose additionally that $f'(0) = 1$ and $f''(0) = 0$. Assume g may be written in wave coordinates as a perturbation of the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1).$$

Writing $T_{\mu\nu} = \epsilon\mathcal{T}_{\mu\nu}$ and discarding terms of $O(\epsilon^2)$, derive the linearised field equations and show that they agree with the linearized Einstein equations in wave gauge

$$\partial^\rho\partial_\rho\overline{h}_{\mu\nu} = -16\pi\mathcal{T}_{\mu\nu}, \qquad \partial_\mu\overline{h}^\mu{}_\nu = 0,$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h_\tau{}^\tau\eta_{\mu\nu}$, and indices are raised and lowered with the Minkowski metric. What does this mean for the predictions of this theory in the weak field regime?

You may assume that in any coordinate basis the Ricci tensor may be written

$$R_{\sigma\nu} = -\frac{1}{2}g^{\mu\rho}\partial_\mu\partial_\rho g_{\sigma\nu} + \Gamma_{\lambda\tau\nu}\Gamma^{\lambda\tau}{}_\sigma + \Gamma_{\lambda\tau\nu}\Gamma^\tau{}_\sigma{}^\lambda + \Gamma_{\lambda\tau\sigma}\Gamma^\tau{}_\nu{}^\lambda$$
$$+\frac{1}{2}\partial_\sigma\Gamma_{\mu\nu}{}^\mu + \frac{1}{2}\partial_\nu\Gamma_{\mu\sigma}{}^\mu - \Gamma_{\mu\lambda}{}^\mu\Gamma_\nu{}^\lambda{}_\sigma$$

and that the wave coordinate condition takes the form $\Gamma_\mu{}^{\nu\mu} = 0$.

4 Let $M = \mathbb{R}_t \times (0, \infty)_r \times (0, \pi)_\theta \times (0, 2\pi)_\phi$ and consider the Schwarzschild metric in Painlevé–Gullstrand coordinates

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\sqrt{\frac{2m}{r}} dr dt + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (*)$$

a) Show that an orthonormal basis of one-forms for this metric is given by $\{e^\mu\}_{\mu=0}^3$ where

$$e^0 = dt, \qquad e^1 = dr + \sqrt{\frac{2m}{r}} dt, \qquad e^2 = r d\theta, \qquad e^3 = r \sin\theta d\phi.$$

b) Find the connection one-forms $\omega^\mu{}_\nu$ associated to this tetrad.

c) We say that a metric has *diagonal curvature operator* with respect to an orthonormal basis $\{f^\mu\}_{\mu=0}^3$ if the curvature two-forms satisfy $\Theta^{\mu\nu} \propto f^\mu \wedge f^\nu$. Show that for any such metric the Ricci tensor must be diagonal with respect to the basis $\{f^\mu\}_{\mu=0}^3$.

d) By computing the curvature two-forms for the metric given in $(*)$, show that it has diagonal curvature operator with respect to the basis $\{e^\mu\}_{\mu=0}^3$ and hence find the Ricci tensor.

e) With reference to the geodesic deviation equation, explain briefly why an observer that approaches the set $\{r = 0\}$ is unlikely to survive the encounter.

You may assume without proof Cartan's first and second structure equations:

$$de^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0, \qquad d\omega^\mu{}_\nu + \omega^\mu{}_\sigma \wedge \omega^\sigma{}_\nu = \Theta^\mu{}_\nu.$$

END OF PAPER