UNIVERSITY OF CAMBRIDGE

MAMA/307, NST3AS/307, MAAS/307

MAT3

MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 1:30 pm to 3:30 pm

PAPER 307

SUPERSYMMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The generators of the Poincaré algebra are the hermitian operators $M^{\mu\nu}$ and P^σ with algebra

$$\begin{aligned}
\left[P^\mu\ ,\ P^\nu\right] &= 0, \\
\left[M^{\mu\nu}\ ,\ P^\sigma\right] &= i\left(P^\mu\,\eta^{\nu\sigma}\ -\ P^\nu\,\eta^{\mu\sigma}\right), \\
\left[M^{\mu\nu}\ ,\ M^{\rho\sigma}\right] &= i\left(M^{\mu\sigma}\,\eta^{\nu\rho}\ +\ M^{\nu\rho}\,\eta^{\mu\sigma}\ -\ M^{\mu\rho}\,\eta^{\nu\sigma}\ -\ M^{\nu\sigma}\,\eta^{\mu\rho}\right).
\end{aligned}$$

A spinor representation of the Lorentz algebra is provided by

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} := \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta},$$

where $\sigma^\mu_{\alpha\dot{\alpha}} = (I, \sigma^1, \sigma^2, \sigma^3)_{\alpha\dot{\alpha}}$, i.e. the 2 by 2 identity matrix followed by the three Pauli matrices, and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (I, -\sigma^1, -\sigma^2, -\sigma^3)^{\dot{\alpha}\alpha}$.

(a) In one sentence, describe what the Coleman-Mandula theorem says about extensions of the Poincaré algebra in quantum field theory. Briefly discuss the assumption that can be weakened to allow supersymmetry, including a short description of the type of algebra that results.

(b) What does the Coleman-Mandula theorem imply for the algebra between an internal symmetry generator T_a and a supersymmetry generator Q_α? Ignoring $R-$symmetries, extend the Poincaré algebra with the $N = 1$ generators Q_α, $\bar{Q}_{\dot{\alpha}}$, giving detailed arguments for the chosen form of each relation.

(c) Consider the action under a parity reversal operator $\hat{P}$, where $|\eta_P| = 1$ and

$$\hat{P}\,Q_\alpha\,\hat{P}^{-1}\ =\ \eta_P\,(\sigma^0)_{\alpha\dot{\beta}}\,\bar{Q}^{\dot{\beta}}, \qquad \hat{P}\,\bar{Q}^{\dot{\alpha}}\,\hat{P}^{-1}\ =\ -\eta_P^*\,(\bar{\sigma}^0)^{\dot{\alpha}\beta}\,Q_\beta.$$

By performing a parity transformation on each side of the algebra between Q_α and $\bar{Q}_{\dot{\beta}}$ that you derived in part (b), check that the relation is consistent with a parity transformation.

(d) Define the *fermion number operator* $(-)^F$ by its action on bosonic states $|B\rangle$ and fermionic states $|F\rangle$. Prove that the number of fermionic states n_F is equal to the number of bosonic states n_B in any supermultiplet. Where does your argument break down for theories that incorporate additional supersymmetry breaking terms in the Lagrangian?

2

(a) Define what it means for a superfield $\Phi(x, \theta, \bar{\theta})$ to be *chiral*. Give an example of a chiral superfield in the MSSM and list the physical component fields contained within it.

Show that, when written in terms of $y^\mu := x^\mu + i\theta\sigma^\mu\bar{\theta}$, a chiral superfield has a particularly simple expansion in terms of component fields. Hence deduce that

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= \phi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x) + A\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \\ &\quad B(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + C(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\phi(x),\end{aligned}$$

where A, B and C are constants which you should determine.

(b) Consider a globally $N = 1$ supersymmetric theory with only three chiral superfields X_{2+}, X_0, X_{2-} where the suffix labels the charge of the superfield under a gauged $U(1)$ symmetry of the theory. Do you expect the theory to possess a gauge anomaly? Explain your answer including Feynman diagrams and a mathematical expression.

Write down the most general superpotential of the model.

Now set any parameters multiplying quadratic or linear superpotential terms to zero. State in one sentence why we expect quantum corrections to respect these conditions.

Write down the scalar potential of the model including any Fayet-Iliopoulos terms, and by analysing it, determine whether and how the model spontaneously breaks supersymmetry, or $U(1)$ symmetry, or both.

[You should use the conventions followed by the course, e.g. $\epsilon^{12} = -\epsilon_{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}} = 1$ and $\eta^{\mu\nu}$ is the 'mostly minus' metric. You may find the following super-covariant derivative useful: $\bar{\mathcal{D}}_{\dot{\alpha}} := -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu$ along with $\{\sigma^\mu, \bar{\sigma}^\nu\} = 2\eta^{\mu\nu}I_2$, where I_2 is the 2 by 2 identity matrix, and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}}$.]

3 Describe supersymmetric gauge unification. You should include graphs depicting the evolution of gauge couplings for both the MSSM and the Standard Model and Feynman diagrams for a contribution to the evolution of g_2 from a superpartner and one from its Standard Model counterpart, but you need not do any explicit computation.

The one-loop evolution of each MSSM gauge coupling $g_a(\mu)$ with renormalisation scale μ is given by

$$\mu \frac{dg_a(\mu)}{d\mu} = \beta_a g_a^3(\mu) \qquad \text{(no sum on } a\text{)},$$

where β_a are real numerical constants. Solve the equation, finding $g_a(\mu)$ in terms of input data $g_a(\mu_0)$.

$g_1(M_Z)$ and $g_2(M_Z)$ are determined very accurately by experiments and can be used, assuming MSSM gauge unification, to provide a prediction of $g_3(M_Z)$. Determine the gauge unification scale M_{GUT} in terms of M_Z, $g_2(M_Z)$ and $g_1(M_Z)$.

Then derive the gauge unification prediction

$$g_3^{-2}(M_Z) = g_2^{-2}(M_Z) + A[g_2^{-2}(M_Z) - g_1^{-2}(M_Z)],$$

where A is a numerical constant which you should evaluate in terms of the β_a, which you may assume all take different values. Do you expect this relation to be approximately satisfied by current measurements? Give your reasoning.

END OF PAPER