UNIVERSITY OF CAMBRIDGE

MAMA/306, NST3AS/306, MAAS/306

MAT3
MATHEMATICAL TRIPOS Part III

Friday 7 June 2024 1:30 pm to 4:30 pm

PAPER 306

STRING THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **Question 1** and **TWO** other questions.
There are **FOUR** questions in total.
All questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 At genus zero, the scattering amplitude for four tachyons in the closed bosonic string is given by

$$\mathcal{A}^{(4)} = \frac{g_s^2}{\mathrm{Vol}(SL(2;\mathbb{C}))} \int DX \; e^{-S[X]} \prod_{i=1}^{4} V(p_i)\,,$$

where $S[X]$ is the gauge-fixed Polyakov action

$$S[X] = \frac{1}{2\pi\alpha'} \int d^2z \, \eta_{\mu\nu} \, \partial X^\mu \bar{\partial} X^\nu \,.$$

and $V(p_i) = \int d^2 z_i \, {:}e^{ip_i \cdot X}{:} \, (z_i, \bar{z}_i)$ is the tachyon vertex operator. The momenta $p_{i\mu}$ obey the mass-shell condition $p_i^2 = 4/\alpha'$.

i) Explain how the dependence of string scattering amplitudes on the string coupling g_s arises in the path integral formulation of the closed string.

ii) Compute the propagator for the scalar fields $X^\mu(z, \bar{z})$.

iii) Hence show that the path integral reduces to

$$\mathcal{A}^{(4)} \sim \frac{g_s^2 \, \delta^{26}\left(\sum_i p_i\right)}{\mathrm{Vol}(SL(2;\mathbb{C}))} \int \prod_{i=1}^{4} d^2 z_i \prod_{j<k} |z_j - z_k|^{\alpha' p_j \cdot p_k} \,.$$

up to numerical factors.

iv) Show that the remaining integral is invariant under $SL(2;\mathbb{C})$ transformations acting as

$$z \mapsto \frac{az+b}{cz+d}\,,$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

The genus zero scattering amplitude for scattering massless states of the Type II superstring includes a factor

$$g_s^2 \frac{\Gamma(-\alpha' s/4)\,\Gamma(-\alpha' t/4)\,\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\,\Gamma(1+\alpha' t/4)\,\Gamma(1+\alpha' u/4)}\,,$$

where s, t, u are the usual Mandelstam variables. $\Gamma(z)$ is nowhere zero and its only singularities are simple poles at $z = 0, -1, -2, \ldots$.

v) By analysing the poles in the s-channel, what can you deduce about the spectrum of the Type II superstring?

2 Consider the theory of a single bosonic scalar field $X : \Sigma \to \mathbb{R}$ on a worldsheet cylinder $\Sigma \cong \mathbb{R} \times S^1$. The worldsheet Hamiltonian H and worldsheet momentum operator P are given by

$$H = \frac{1}{2}\hat{p}^2 - \frac{1}{12} + \sum_{n=1}^{\infty} (\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n)$$

$$P = \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}\tilde{\alpha}_n - \alpha_{-n}\alpha_n) \ ,$$

where the operators $\hat{p}$, α_n, $\tilde{\alpha}_n$ obey commutation relations $[\alpha_n, \tilde{\alpha}_m] = [\alpha_m, \hat{p}] = [\tilde{\alpha}_m, \hat{p}] = 0$ while $[\alpha_n, \alpha_m] = [\tilde{\alpha}_n, \tilde{\alpha}_m] = n\,\delta_{n,-m}$.

i) Briefly explain the interpretation of each term in the Hamiltonian.

ii) Give a basis of eigenstates of H and P and compute their eigenvalues in this basis.

iii) The theory is now placed on a worldsheet torus $\mathbb{C}/\Lambda$ where $\Lambda = \{(2\pi k_1, 2\pi\tau k_2)\}$ for $k_1, k_2 \in \mathbb{Z}$ and $\mathrm{Im}(\tau) > 0$. Show that the partition function

$$Z(\tau, \bar{\tau}) = \frac{V}{2\pi} \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \, |\eta(\tau)|^{-2} \, ,$$

where V is the (infinite) volume of the target space, and where you should define the function $\eta(\tau)$.

Now consider the case that the target space is a circle of radius R, so that $X \sim X + 2\pi R$.

iv) What periodicity conditions must be demanded of $X(\sigma^\alpha)$ on a worldsheet cylinder? How does this affect the Hamiltonian and momentum operators H and P above?

v) Compute the partition function for the theory with this target space.

3 Consider the theory with action

$$S[X,\psi] = \frac{1}{2}\int_{\Sigma}\left(\partial_z X \partial_{\bar z} X + \psi\partial_{\bar z}\psi + \bar\psi\partial_z\bar\psi\right)\, d^2 z\,,$$

where X is a bosonic scalar while ψ and $\bar\psi$ are fermionic fields. The non-trivial OPEs of this theory are

$$X(z,\bar z)X(w,\bar w) \sim -\ln|z-w|^2\,, \qquad \psi(z)\psi(w) \sim \frac{1}{z-w}\,, \qquad \bar\psi(\bar z)\bar\psi(\bar w) \sim \frac{1}{\bar z - \bar w}\,,$$

while the stress tensor of the theory has non-zero components

$$\begin{aligned} T(z) &= -\frac{1}{2}:\partial_z X\partial_z X: \,-\frac{1}{2}:\psi\partial_z\psi: \\ \bar T(\bar z) &= -\frac{1}{2}:\partial_{\bar z} X\partial_{\bar z} X: \,-\frac{1}{2}:\bar\psi\partial_{\bar z}\bar\psi: \ . \end{aligned}$$

i) What does it mean for an operator $\mathcal{O}(z,\bar z)$ in a 2d CFT to be *primary* and have *weights* $(h,\tilde h)$? How are the weights related to the spin and scaling dimension of the operator?

ii) Explain the meaning of the normal ordering symbols in T and $\bar T$.

iii) Show that ψ is a primary operator and find its weights.

iv) Calculate the (holomorphic) central charge c of this theory.

v) The current G is defined by $G = i\psi\,\partial_z X$. Show that G is holomorphic when the fields obey their equations of motion. Compute the $G(z)G(w)$ OPE.

vi) Given that G has a mode expansion

$$G(z) = \sum_{m\in\mathbb{Z}} \frac{G_m}{z^{m+\frac{3}{2}}}\,,$$

show that the GG OPE implies that the modes obey anticommutation relations

$$\{G_m, G_n\} = 2L_{m+n} + \frac{\mathrm{c}}{12}(4m^2-1)\,\delta_{m,-n}$$

where L_m are the modes of the stress tensor. *[You may assume the anticommutator of modes of the fermionic current G involves the same combination of contour integrals as the commutator of modes of a bosonic current.]*

4

i) Write down the worldsheet action for a closed bosonic string propagating in a 26-dimensional space-time on which there is a background metric $G_{\mu\nu}$, Kalb-Ramond field $B_{\mu\nu}$ and dilaton Φ.

ii) Without detailed calculation, explain why the theory is consistent only when these background fields obey

$$0 = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_\mu{}^{\kappa\lambda} H_{\nu\kappa\lambda} + \mathcal{O}(\alpha')$$
$$0 = -\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} + \nabla^\lambda \Phi \, H_{\lambda\mu\nu} + \mathcal{O}(\alpha')$$
$$0 = 2\nabla^2\Phi - 2\nabla^\mu\Phi\nabla_\mu\Phi + \frac{1}{2}R - \frac{1}{24}H^{\lambda\mu\nu}H_{\lambda\mu\nu} + \mathcal{O}(\alpha')\,.$$

iii) Show that if the first two equations in part ii) hold, the right hand side of the third equation is necessarily constant (ignoring terms of order α'). *[You may assume without proof the Bianchi identities*

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \qquad \textit{and} \qquad H^{\lambda\mu\nu}\nabla_\lambda H_{\kappa\mu\nu} = \frac{1}{6}\nabla_\kappa (H^{\lambda\mu\nu}H_{\lambda\mu\nu})$$

for the Ricci curvature and H-field.]

iv) Give two different reasons why the above equations are expected to receive corrections at higher order in α'.

v) Now consider an open string, whose worldsheet has the topology of an infinite strip, and whose endpoints are attached to a Dp-brane. Suppose this string travels through a region where the background fields $(G_{\mu\nu}, B_{\mu\nu}, \Phi) = (\eta_{\mu\nu}, b_{\mu\nu}, \phi)$, where η is the Minkowski metric and b and ϕ are constant. What boundary conditions must the string obey? How is the background B-field interpreted from the perspective of the Dp-brane?

END OF PAPER