UNIVERSITY OF CAMBRIDGE

MAMA/305, NST3AS/305, MAAS/305

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2024 1:30 pm to 4:30 pm

PAPER 305

THE STANDARD MODEL

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A left-handed Weyl spinor ψ_L and a right-handed Weyl spinor ψ_R are governed by the action

$$S = -\int d^4x \, \left(i\bar{\psi}_L\bar{\sigma}^\mu\partial_\mu\psi_L + i\bar{\psi}_R\sigma^\mu\partial_\mu\psi_R - m\bar{\psi}_R\psi_L - m^\star\bar{\psi}_L\psi_R\right)$$

with $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$ and $m \in \mathbb{C}$.

(i) Find an action of parity $P : \mathbf{x} \mapsto -\mathbf{x}$ on the spinors that leaves the action invariant.

(ii) Both spinors are subsequently coupled to a gauge field A_μ with charge $+1$. Find an action of charge conjugation C on the gauge field and spinors that leaves the equations of motion invariant.

(iii) The spinors interact with a complex scalar ϕ through the Yukawa couplings

$$S_{\text{Yuk}} = -\int d^4x \, \left(\lambda_1 \, \phi\psi_L\psi_L + \lambda_2 \, \phi\psi_R\psi_R + \text{h.c.}\right)$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$. By considering the most general actions of P and C on the scalar ϕ and the spinors, determine the conditions on λ_1 and λ_2 so that the theory is invariant (a) under P and (b) under C.

(iv) The fermions interact through the couplings

$$S_{4-\text{fermi}} = \int d^4x \, \left(g\bar{\psi}_L\bar{\sigma}^\mu\psi_L + g'\bar{\psi}_R\sigma^\mu\psi_R\right)\left(g\bar{\psi}_L\bar{\sigma}_\mu\psi_L + g'\bar{\psi}_R\sigma_\mu\psi_R\right)$$

with $g, g' \in \mathbb{R}$. What are the dimensions of g and g'? What are the conditions on g and g' so that this interaction respects parity?

2 A gauge theory has coupling g^2 which depends on the scale μ through the formula

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{b_0}{3(4\pi)^2} \log \frac{\Lambda_{UV}^2}{\mu^2}$$

Here Λ_{UV} is the UV cut-off and b_0 is the coefficient of the one-loop beta function. For an $SU(N_c)$ gauge theory, coupled to N_f Weyl fermions (either left-handed or right-handed) and N_s scalars, all in the fundamental representation, the coefficient is given by

$$b_0 = 11N_c - N_f - \frac{1}{2}N_s$$

(i) What does it mean for the theory to be *asymptotically free*? What is the condition on N_f, N_s and N_c for asymptotic freedom?

(ii) If the theory is asymptotically free, define the strong coupling scale Λ. (For QCD, this is the scale that is usually referred to as Λ_{QCD}.)

(iii) What is the maximum number of generations N_g of the Standard Model for which the $SU(3)$ gauge group is asymptotically free? What is the maximum number of generations for which the $SU(2)$ gauge group is asymptotically free?

(iv) Denote the coupling constants for $SU(3)$, $SU(2)$ and $U(1)_Y$ as g_s, g_w and g_Y respectively. Suppose that there exists a scale M for which

$$g_s^2(M) = g_w^2(M) = \frac{5}{3}g_Y^2(M) \qquad (\dagger)$$

Show that the coupling constants measured at some other scale $\mu < M$ must obey

$$\frac{1}{g_s^2(\mu)} - \frac{1}{g_w^2(\mu)} = \frac{b_s - b_w}{\frac{3}{5}b_Y - b_w}\left(\frac{3}{5g_Y^2(\mu)} - \frac{1}{g_w^2(\mu)}\right)$$

where b_s, b_w and b_Y denote the coefficients of the one loop beta function for the respective gauge groups.

(v) Show that $U(1)_Y \times SU(2)_w \times SU(3)_s$ is a subgroup of $SU(5)$. By considering the covariant derivative acting on a field in the fundamental representation of $SU(5)$, explain why one would expect the relationship (†) between couplings.

3 Comment on the meaning and implications of a symmetry which has:

- a gauge anomaly,
- a chiral, or ABJ, anomaly,
- a 't Hooft anomaly.

Consider one generation of the Standard Model, with left- and right-handed Weyl fermions transforming in the following representations of $U(1) \times SU(2) \times SU(3)$:

$$Q_L : (\mathbf{2}, \mathbf{3})_q \,, \quad L_L : (\mathbf{2}, \mathbf{1})_l \,, \quad u_R : (\mathbf{1}, \mathbf{3})_u \,, \quad d_R : (\mathbf{1}, \mathbf{3})_d \,, \quad e_R : (\mathbf{1}, \mathbf{1})_x \,.$$

This is the usual set of representations (ignoring the right-handed neutrino), but with the hypercharges q, l, u, d, and x left arbitrary. Assume that all hypercharges are integers i.e. $q, l, u, d, x \in \mathbb{Z}$

(i) Write down all the conditions for anomaly cancellation for a single generation, *excluding* the mixed gauge-gravitational anomaly.

(ii) Argue that the anomaly relations require $u - d = 2y$ for $y \in \mathbb{Z}$.

(iii) Show that the anomaly conditions require integer solutions to $54q^3 + 18y^2 q + x^3 = 0$.

(iv) Assume that $q \neq 0$. Argue that the anomaly conditions require solutions to $54 + 18\tilde{y}^2 + \tilde{x}^3 = 0$ with $\tilde{x}, \tilde{y} \in \mathbb{Q}$.

(v) Show that there is a unique solution to the anomaly equations (up to scaling). If we choose $q = 1$, what are the resulting hypercharges l, u, d, and x?

[*You may assume, without proof, that there is no non-trivial solution to the equation* $a^3 + b^3 = c^3$, *with* $a, b, c \in \mathbb{Z}$. *You may also find the following change of variables useful:*

$$\tilde{x} = -\frac{6}{v+w} \quad \text{and} \quad \tilde{y} = \frac{3(v-w)}{v+w} \,.]$$

(vi) Show that this solution automatically satisfies the mixed gauge-gravitational anomaly.

4 List the fermions in one generation of the Standard Model, including the right-handed neutrino, and give their representations under the gauge group.

Explain why it's not possible to write down mass terms for the fermions but how, with judiciously chosen quantum numbers for the Higgs boson, it is possible for them all to gain a mass through Yukawa terms.

Explain how the small observed value of the neutrino mass may be due to a large Majorana mass for the right-handed neutrino. How is it possible to capture this same physics in the absence of a right-handed neutrino?

Suppose that the Standard Model is extended to include a complex scalar field ϕ, transforming in the $\mathbf{3}$ of $SU(2)$, a singlet under $SU(3)$, and with hypercharge $Y = +1$. What are the electric charges of the ϕ field after electroweak symmetry breaking?

Write

$$\phi = \frac{1}{2}\begin{pmatrix} \phi^3 & \phi^1 - i\phi^2 \\ \phi^1 + i\phi^2 & -\phi^3 \end{pmatrix}$$

with $\phi^1, \phi^2, \phi^3 \in \mathbb{C}$. Find the components of ϕ that are uncharged under electromagnetism.

Write down a gauge invariant Yukawa coupling between ϕ and the fermions of the Standard Model. Show that if ϕ condenses without breaking electromagnetism then the left-handed neutrino gets a Majorana mass.

END OF PAPER