UNIVERSITY OF CAMBRIDGE

MAMA/304, NST3AS/304, MAAS/304, NST3PHY/AQFT, MAPY/AQFT

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 1:30 pm to 4:30 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Lagrangian for a scalar field in d dimensional Minkowski spacetime is

$$\mathcal{L}_n = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{n!}\phi^n$$

where g is a coupling constant. Consider the generating functional defined by

$$Z_n[J] = \mathcal{N}\int \mathcal{D}\phi \, \exp\left(i\int d^dx\,(\mathcal{L}_n + J(x)\phi(x))\right),$$

where $\mathcal{N}$ is an unspecified constant.

(a) For $n = 5$, draw all leading order connected Feynman diagrams that contribute to the partition function $Z_5[0]$.

(b) For $n = 2$ find an explicit form, in terms of the sources $J(x)$, for the the generating functional $Z_2[J]$. In your answer choose $\mathcal{N}$ so that $Z_2[0] = 1$.

(c) Show that the generating functional for general n is related to the generating functional for $n = 0$ by

$$Z_n[J] = \exp\left(-ig\frac{(-i)^n}{n!}\int d^dx\frac{\delta^n}{\delta J(x)^n}\right) Z_0[J].$$

State any assumptions you make in deriving this result.

(d) Expand the expression for $Z_2[J]$ given in (b) above to the first non-trivial order in g and show that it agrees with the expression in (c).

2

Consider the Lagrangian for the scalar field $\phi(x)$ in d-dimensional Minkowski space

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \frac{1}{2}\delta_Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta_{m^2}\phi^2 - \frac{1}{4!}\delta_\lambda\phi^4.$$

(a) Briefly explain the role of each of the terms in the Lagrangian and, without derivation, write down the momentum space Feynman rules of this theory.

(b) Draw all diagrams up to and including one-loop order for the four-point connected correlation function. Using dimensional regularization, and writing $\lambda = \mu^\epsilon g$ (where g is dimensionless and $d = 4-\epsilon$), show that the one-loop counter-term δ_g in the MS (minimal subtraction) scheme is

$$\delta_g = \frac{3g^2}{16\pi^2\epsilon}.$$

(c) Show that, to leading order

$$\mu\frac{d}{d\mu}\Big(\mu^\epsilon(g+\delta_g) - 2\mu^\epsilon g\delta_Z\Big) = 0$$

and hence calculate the beta-function for λ up to leading order. You may assume, without justification, that there is no wavefunction renormalization at leading order.

[Hint: You may find the following integrals useful

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2}, \qquad \int_{\mathbb{R}^d} \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}.$$

The behaviour of the Gamma function near zero is

$$\Gamma(z) = \frac{1}{z} - \gamma + ...,$$

where γ is the Euler-Mascheroni constant.]

3

The action for a scalar field theory in d dimensional Euclidean space is

$$S[\Phi] = \int d^d x \left(\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{3!} g \Phi^3 \right).$$

The quantum field $\Phi(x)$ is defined with a momentum cut-off Λ.

(a) The Fourier transform of $\Phi(x)$, denoted by $\widetilde{\Phi}(p)$, is decomposed into high and low momentum modes with respect to some arbitrary scale $b\Lambda < \Lambda$, where $0 < b < 1$, as follows

$$\widetilde{\Phi}(p) = \begin{cases} \widetilde{\phi}(p) & \text{if } p^2 < b^2\Lambda^2 \\ \widetilde{\chi}(p) & \text{if } b^2\Lambda^2 < p^2 < \Lambda^2 \end{cases}$$

Show that

$$\Phi(x) = \phi(x) + \chi(x),$$

where $\widetilde{\phi}(p)$ and $\widetilde{\chi}(p)$ are the Fourier transforms of $\phi(x)$ and $\chi(x)$ respectively.

(b) The effective action $W[\phi]$ is given by

$$e^{-W[\phi]} = \int \mathcal{D}\chi \, e^{-S[\Phi]}.$$

Show that $W[\phi]$ can be written in the form

$$e^{-W[\phi]} = e^{-S[\phi]} \Big[\exp\Big(- S_{\text{int}}[\phi, -\delta/\delta J] \Big) Z_\chi[J] \Big]_{J=0},$$

where $Z_\chi[J]$ is the generating functional for a free, massless field $\chi(x)$ and S_{int} is a functional that you should find. Briefly explain how this action can be evaluated in perturbation theory.

(c) Assuming the effective action can be written as

$$W[\phi] = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{3!} g_{\text{eff}} \, \phi^3 + \ldots \right),$$

where $+\ldots$ denote terms in the effective Lagrangian which you may neglect. Draw all Feynman diagrams that you would need in order to evaluate the effective coupling g_{eff} to order g^3. Find g_{eff} to order g^3 and show that, if $d = 6$, then

$$g_{\text{eff}} = g + C g^3 \ln(b) + \ldots,$$

where C is a constant you should specify and b is the constant defined in part (a). You may neglect any effects of wavefunction rescaling.

[Hint: The surface area of a $d-1$ dimensional sphere is $\mathbb{R}^d$ is

$$\frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

It may also be helpful to recall that $\Gamma(n) = (n-1)!$, when $n \in \mathbb{Z}_+$.]

4

The Lagrangian for $SU(N)$ gauge fields $A_\mu(x)$, coupled to fermions $\psi_i(x)$ in the fundamental representation is given by

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}^i(i\partial\!\!\!/ - m)\psi_i + J_a^\mu A_\mu^a,$$

where J_a^μ is a source that is a function of the fermion fields, $i, j = 1, 2, \ldots, N$ and $a, b = 1, 2, \ldots, N^2 - 1$. The fields transform under the gauge symmetry as

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger, \qquad \psi \to U\psi,$$

with

$$U = e^{i\alpha^a(x)T_a},$$

where $\alpha^a(x)$ is the parameter of the gauge transformation. Here and throughout the question you may assume that $A_\mu(x) = A_\mu^a(x)T_a$ and $F_{\mu\nu}(x) = F_{\mu\nu}^a(x)T_a$, where the T_a are generators of $SU(N)$.

(a) From the transformation of A_μ, derive the transformation of the field strength $F_{\mu\nu}$. Hence show that the infinitesimal transformation of $F_{\mu\nu}$ may be written as

$$\delta F_{\mu\nu}^a = f_{bc}{}^a F_{\mu\nu}^b \alpha^c,$$

where $f_{ab}{}^c$ are constants that you should define in terms of the generators of the gauge group.

(b) Assuming that the Lagrangian is gauge-invariant, deduce that

$$J^\mu \to U J^\mu U^\dagger$$

and hence find an expression for J^μ.

(c) Now consider a theory of $SU(N)$ gauge fields coupled to fermions $\Psi(x)$ in the adjoint representation. The fermions transform as

$$\Psi \to U\Psi U^\dagger.$$

Find the infinitesimal version of the transformation of $\Psi(x)$ and derive a gauge-invariant Lagrangian.

(d) Draw the Feynman diagrams that contribute to the gluon propagator up to and including one-loop order. Briefly comment on how the evaluation of these diagrams differs if we have fermions in the fundamental representation (as in parts (a) and (b)) and fermions in the adjoint representation (as in part (c)). You do not have to evaluate the diagrams.

END OF PAPER