UNIVERSITY OF CAMBRIDGE

MAMA/303, NST3AS/303, MAAS/303

MAT3

MATHEMATICAL TRIPOS Part III

Friday 7 June 2024 9:00 am to 11:00 am

PAPER 303

STATISTICAL FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a theory which has an effective free energy

$$f(T,m) = a_4(T)\, m^4 + a_6\, m^6 - B\, m\,,$$

in the mean field approximation, where m is the magnetisation, $a_6 > 0$, $a_4(T)$ varies from positive to negative as the temperature T is lowered and $a_4(T) \sim (T - T_c)$ close to $T = T_c$.

(i) For given a_4, a_6, B and T, how can the equilibrium value of m be determined from f?

(ii) If $B = 0$ show that there is a phase transition at $T = T_c$. Is this a continuous or discontinuous phase transition? Compute the critical exponents α, β, γ and δ at this phase transition. [You should compute γ for both $T \to T_c^+$ and $T \to T_c^-$.] [*Hint: Recall that close to the critical point* $c \sim |T - T_c|^{-\alpha}$, $m \sim (T_c - T)^{\beta}$ *for* $T < T_c$, $m \sim |B|^{1/\delta}$ *and* $\chi \sim |T - T_c|^{-\gamma}$, *where* c *is the heat capacity and* χ *is the magnetic susceptibility.*]

(iii) Does the system possess another phase transition? If so, is it continuous or discontinuous? Justify your answers.

(b) Consider a model defined on a square lattice in d dimensions. The N lattice sites are labeled by i and the spin variable $\boldsymbol{\sigma}_i = (\cos\theta_i, \sin\theta_i)$ at site i is a 2-dimensional vector that can point in p different directions, where $\theta_i = 2\pi n_i/p$ and $n_i = 0, 1, \dots p-1$. The energy is given by

$$E = -J \sum_{\langle ij \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\,,$$

where $J > 0$ is a constant and $\langle ij \rangle$ means that the sum is over nearest neighbour pairs.

(i) For the model above with $p = 4$, use the mean field approach to show that the effective free energy per unit site can be approximated by

$$f = \frac{F}{N} = A\, m^2 - T \ln\Big[C + D\cosh\big(\beta J q m\big)\Big]\,,$$

where $m = |\boldsymbol{m}|$ is the magnitude of the magnetisation, $\beta = 1/T$, q is the number of nearest neighbours of each site, and you should determine A, C and D. [*Hint: Use the approach where* $\boldsymbol{\sigma}_i$ *is written as* $\boldsymbol{\sigma}_i = \boldsymbol{m} + \delta\boldsymbol{\sigma}_i$ *and terms of order* $(\delta\sigma_i)^2$ *and higher are neglected.*]

(ii) From the expression for f in part (b)(i), find an implicit equation for the equilibrium value of m. [You should keep $p = 4$ here and in all subsequent parts of this question unless otherwise stated.]

(iii) Now, treat m^2 as small and expand f as a power series in m^2, neglecting m^6 and higher terms. For what value of T is there a phase transition?

(iv) What would you expect the lower critical dimension for this model to be? What happens to the model in the limit $p \to \infty$ and what would you expect the lower critical dimension to be in that case? Briefly justify your answers.

2

Consider a theory involving a real scalar field ϕ in d dimensions with a free energy of the form

$$F[\phi] = \int d^d x \Big[\tfrac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + \tfrac{1}{2}\mu_0^2 \, \phi^2 + \ldots \Big] . \qquad (*)$$

(a) Describe the three steps of the renormalisation group procedure (in momentum space) for such a free energy and explain how they result in a flow of the parameters in the free energy. You should denote the original cutoff Λ and the new cutoff Λ/ζ.

(b) Calculate the naive (engineering) dimensions of ϕ and μ_0^2.

(c) Why can the scaling dimension Δ_ϕ of the field ϕ differ from the engineering dimension?

(d) Now suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with $\sim \int d^d x \, \alpha \, \nabla^{2m} \phi^{2n}$, where $n > 1$ and $m \geqslant 0$ are integers. Compute the naive (engineering) dimension of the coupling α. Give conditions on d in terms of n and m for the coupling to be relevant, marginal and irrelevant.

(e) Now instead suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with

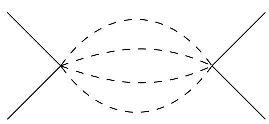
$$\sim \int d^d x \Big[g_0 \, \phi^4 + \gamma_0 \, \phi^5 + \lambda_0 \, \phi^6 \Big] .$$

(i) Draw a Feynman diagram that represents a contribution that gives rise to a non-zero anomalous dimension of ϕ at order g_0^2. Justify you answer. [You are not required to compute the correction, but you should explain how the diagram you have drawn gives a non-zero anomalous dimension.]

(ii) Ignoring other interactions, does the $\lambda_0 \phi^6$ term alone give a correction to the flow of the coupling $\gamma(\zeta)$. If so, draw a Feynman diagram that represents such a contribution; if not, explain why.

(iii) Draw a Feynman diagram representing the correction to the flow of the coupling $g(\zeta)$ from the $\lambda_0 \phi^6$ term at order λ_0. Calculate this contribution. [Here and in all subsequent parts of this question you may assume that $\langle \phi^+_{\boldsymbol{k}} \, \phi^+_{\boldsymbol{k}'} \rangle_+ = (2\pi)^d \, \delta^{(d)}(\boldsymbol{k} + \boldsymbol{k}') G_0(k)$, where $G_0(k) = 1/(k^2 + \mu_0^2)$, for appropriately defined ϕ^+ and $\langle \ldots \rangle_+$, you may use Wick's theorem without proof, and you may leave your final answer in integral form. You may ignore other corrections, including those from the rescaling of the field.]

(iv) Calculate the contribution to $g(\zeta)$ at order λ_0^2 represented by the following Feynman diagram.



(v) Draw Feynman diagrams representing all other corrections to $g(\zeta)$ from the $\lambda_0 \phi^6$ term at order λ_0^2. [You do not need to calculate these contributions.]

3

Consider an $O(N)$ model involving an N-component real field $\boldsymbol{\phi}(\boldsymbol{x})$ in d dimensions with free energy,

$$F[\boldsymbol{\phi}] = \int d^d x \Big[\tfrac{1}{2} (\partial_i \phi_a)(\partial_i \phi_a) + \tfrac{1}{2} \sum_a \mu_{0,a}^2 \, \phi_a^2 + g_0 \, (\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2 \Big] \, ,$$

where $g_0 > 0$, $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = \phi_a \phi_a$, repeated indices are summed over, $i = 1, 2, \ldots, d$ and $a = 1, 2, \ldots, N$.

(a) Suppose that $\mu_{0,a}^2 = \mu_0^2$ for all a and $\mu_0^2 \sim (T - T_c)$.

(i) What is the symmetry of F? What is the symmetry of the mean field theory ground state when $\mu_0^2 > 0$ and when $\mu_0^2 < 0$?

(ii) Draw Feynman diagrams representing the leading order corrections to μ_0^2 from the g_0 term. Indicate which diagrams (if any) are proportional to N.

(iii) Draw Feynman diagrams representing the leading order corrections to g_0 from the g_0 term. Indicate which diagrams (if any) are proportional to N.

(iv) The leading order corrections to the couplings are

$$\mu^2(\zeta) = \zeta^A \left[\mu_0^2 + 4(N+2) g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \right] ,$$

$$g(\zeta) = \zeta^B \left[g_0 - 4(N+8) g_0^2 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2} \right] . \qquad (*)$$

Determine the constants A and B.

(v) Derive the β functions, $\frac{d\mu^2}{ds}$ and $\frac{dg}{ds}$, where $s = \ln \zeta$. [You may use Ω_{d-1} to denote the area of the unit sphere S^{d-1}.]

(vi) Consider $d = 4 - \epsilon$ and define $\tilde{g} = \Lambda^{-\epsilon} g$. Working to leading order in ϵ, find the fixed points and calculate the critical exponent ν at each fixed point.

(b) Now consider $N = 2$, and allow $\mu_{0,1}^2$ and $\mu_{0,2}^2$ to differ.

(i) Assuming mean field theory, describe the phase diagram in the $\mu_{0,1}^2, \mu_{0,2}^2$ plane. Identify points/lines where phase transitions occur and describe the nature of any phase transitions.

(ii) What are the equations for $\mu_1^2(\zeta)$ and $\mu_2^2(\zeta)$ in this theory analogous to the equation for $\mu^2(\zeta)$ in $(*)$?

(iii) Is a term $\sim \phi_1 \phi_2$ generated by interactions? Briefly justify your answer.

END OF PAPER