UNIVERSITY OF CAMBRIDGE

MAMA/302, NST3AS/302, MAAS/302

MAT3

MATHEMATICAL TRIPOS Part III

Monday 3 June 2024 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For each of the following sets and binary operations, determine whether they form a group. If it is a group, show that the group axioms are satisfied; if not, give a group axiom which is not satisfied.

(i) The integers under addition, $(\mathbb{Z}, +)$.

(ii) The nonzero integers under multiplication, $(\mathbb{Z}_*, \times)$.

(iii) The subset of linear functions $\mathcal{L}$ under composition $\circ$, where

$$\mathcal{L} = \{\, f : \mathbb{R} \to \mathbb{R} \mid f(x) = ux + b;\; b \in \mathbb{R};\; u \in \mathbb{R}, u \neq 0 \,\} \,.$$

(b) Consider the following matrix Lie group

$$G = \left\{ \begin{pmatrix} 1 & q & s \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, q, r, s \in \mathbb{R} \right\} .$$

[The group operation is matrix multiplication.]

(i) Find the centre of the group, $Z(G)$.

(ii) Find the Lie algebra of the group, $L(G)$. Choose a basis for $L(G)$ and determine the corresponding structure constants.

(iii) Show that the exponential map, $\exp : L(G) \to G$, is both one-to-one and onto. Explain whether G is connected.

(iv) Explain whether $L(G)$ is simple, semisimple, or neither.

(c) Consider the inner product space (in fact, Hilbert space) of square-integrable, complex functions

$$V = \{\, \psi : \mathbb{R} \to \mathbb{C} \mid (\psi, \psi) = 1 \,\} \,,$$

where the inner product of two functions in the space is defined to be

$$(\phi, \psi) = \int_{-\infty}^{\infty} dx\, (\phi(x))^* \psi(x) \,.$$

Taking the group G to be the same as in (b), consider a map $D : G \to GL(V)$ such that

$$\forall g = \begin{pmatrix} 1 & q & s \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \in G \,, \quad D(g)\, \psi(x) = e^{-is} e^{irx} \psi(x - q) \,.$$

Show that D is a unitary representation of G. If $\psi(x)$ represents a time-independent quantum wavefunction in one spatial dimension, how can this transformation be interpreted physically?

2

(a) Given a Lie algebra $\mathfrak{g}$, state the defining properties of the Lie bracket and the structure constants of $\mathfrak{g}$.

(b) Let G be a Lie group and let $L(G)$ be its Lie algebra.

(i) Define the adjoint representation of G, $\mathrm{Ad} : G \to GL(L(G))$, and confirm that it is indeed a representation.

(ii) From the adjoint representation of G, derive the adjoint representation of $L(G)$, $\mathrm{ad} : L(G) \to \mathfrak{gl}(L(G))$.

(c) Take as given that the Killing form of a Lie algebra $\mathfrak{g}$ is the symmetric, bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ such that

$$\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)\,,$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Given a basis for $\mathfrak{g}$, $\{\, T_a \,\}$ with $a = 1, \ldots, \dim \mathfrak{g}$, derive an expression for $\kappa(T_a, T_b)$ in terms of the structure constants.

(d) Take the following as a basis for $\mathfrak{su}(2)$

$$\tau_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \tau_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \tau_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

Determine the Killing form $\kappa(\tau_a, \tau_b)$ and explain whether this is an adapted basis.

(e) Explain what is meant by the complexification of $\mathfrak{su}(2)$, that is by $\mathfrak{su}(2)_{\mathbb{C}}$. Determine the Killing form using the following basis

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

and explain whether this is an adapted basis.

(f) Take as a basis for $\mathfrak{gl}(n, \mathbb{C})$ the $n \times n$ matrices $\{\, \mathsf{T}^i{}_j \,\}$, where for each $\mathsf{T}^i{}_j$, the matrix element in the i-th row and j-th column is equal to 1, and all other matrix elements are 0. Let $X = \mathsf{T}^i{}_j X^j{}_i$ and $Y = \mathsf{T}^i{}_j Y^j{}_i \in \mathfrak{gl}(n, \mathbb{C})$. Derive an expression for $\kappa(X, Y)$ in terms of the components of X and Y.

[*You may use without proof the relation* $[\mathsf{T}^i{}_j, \mathsf{T}^k{}_\ell] = \delta^k{}_j \mathsf{T}^i{}_\ell - \delta^i{}_\ell \mathsf{T}^k{}_j$.]

3

This question concerns the symmetries of special relativity in 1 temporal and 3 spatial dimensions.

(a) Define the transformations which comprise the Lorentz and Poincaré groups. Explain how the Lorentz and Poincaré groups are related mathematically, justifying your answer.

The basis elements of the Poincaré algebra are $M^{\mu\nu}$ and P^{σ}. These satisfy the following:

$$
\begin{aligned}
M^{\nu\mu} &= -M^{\mu\nu} \\
[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho} \\
[M^{\mu\nu}, P^{\sigma}] &= \eta^{\nu\sigma} P^{\mu} - \eta^{\mu\sigma} P^{\nu} \\
[P^{\mu}, P^{\nu}] &= 0 \,,
\end{aligned}
$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric.

(b) Among these basis elements, identify the generators of spatial rotations J_1, J_2, and J_3, and the generators of Lorentz boosts K_1, K_2, K_3. Use the Lie brackets given above to determine $[J_1, J_2]$, $[J_1, K_2]$, and $[K_1, K_2]$.

The universal enveloping algebra contains the Pauli-Lubanski pseudovector

$$W_{\mu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^{\sigma} \,.$$

(c) Show that $W_{\mu} P^{\mu} = 0$ and that $[W_{\mu}, P^{\tau}] = 0$.

(d) In its own rest frame, a particle of mass $m > 0$ is in a spin state $|\psi\rangle = |j, j_3\rangle$, where $2j$ is a nonnegative integer and $2j_3$ is an integer with $-j \leqslant j_3 \leqslant j$. Show that $|\psi\rangle$ is an eigenstate of W_0 and W_3 and find the corresponding eigenvalues.

4

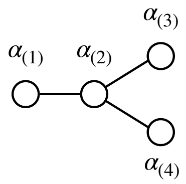
In this question you may use without proof any general results from the theory of complex, simple, finite-dimensional Lie algebras and their representations provided they are clearly stated.

(a) With regard to complex, simple, finite-dimensional Lie algebras, give brief descriptions of the following concepts:

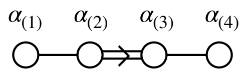
- (i) Cartan subalgebra
- (ii) Root set
- (iii) Root string
- (iv) Cartan matrix

(b) For each of the two Dynkin diagrams below, write down the corresponding Cartan matrix, paying attention to how the nodes are labelled.

(i)



(ii)



(c) Show that there are exactly N distinct complex, simple Lie algebras of rank 2, where you should determine N.

(d) Consider the Lie algebra $\mathfrak{g}$ specified by the Cartan matrix

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} ,$$

and denote its simple roots by α and β, where $|\alpha| > |\beta|$. Determine the full root set of $\mathfrak{g}$ and the dimension of $\mathfrak{g}$. What is the angle between α and β?

(e) Find the fundamental weights

$$\chi = C_1\alpha + C_2\beta$$
$$\omega = \alpha + C_3\beta$$

where C_1, C_2 and C_3 are numbers you should determine. Find the irreducible representation whose highest weight is ω.

(f) Consider the $\mathfrak{su}(2)_{\mathbb{C}}$ subalgebras of $\mathfrak{g}$ associated with the simple roots α and β, denoted $\mathfrak{sl}(2)_\alpha$ and $\mathfrak{sl}(2)_\beta$, respectively. Show explicitly that the adjoint representation of $\mathfrak{g}$ can be written as a direct sum of irreducible representations of $\mathfrak{sl}(2)_\alpha$ and also of $\mathfrak{sl}(2)_\beta$. [*Recall that we labelled the irreducible representations of* $\mathrm{su}(2)_{\mathbb{C}}$ *by* d_Λ, *with* Λ *the highest weight for the irrep.*]

END OF PAPER