UNIVERSITY OF CAMBRIDGE

MAMA/301, NST3AS/301, MAAS/301, NST3PHY/QFT, MAPY/QFT

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Proca Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \ ,$$

which describes a massive gauge field. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $m^2 > 0$.

(a) Verify that this theory is not invariant under a standard gauge transformation of A_μ. Show that the equations of motion imply that

$$\left(\partial_\mu \partial^\mu + m^2\right) A_\nu = 0 \ , \quad \partial_\mu A^\mu = 0 \ .$$

(b) Evaluate the energy-momentum tensor $T_{\mu\nu}$ for the Proca Lagrangian. Show that, on-shell, the energy density is positive definite, up to a total spatial derivative.

(c) Find the plane-wave solutions to the equations in part (a). Show that there are 3 polarisation states.

(d) Construct the propagator for the Proca field, that is, give an integral expression for

$$\Delta_{\mu\nu}(x-y) = \langle \Omega | \mathrm{T}\left(A_\mu(x) A_\nu(y)\right) | \Omega \rangle \ .$$

[*Hint: Recall that the propagator is a Green's function. Write the equation of motion for* A_μ *as* $\Pi_{\mu\nu} A^\nu = 0$*, then define an appropriate inverse for the operator* $\Pi_{\mu\nu}$*.*]

2 Consider the following Lagrangian for a real scalar field in four dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \ ,$$

where m and g are real positive constants.

(a) What is the mass dimension of g? Briefly justify your answer.

(b) What are the Feynman rules for a theory described by the Lagrangian $\mathcal{L}$? A brief (but clear) explanation is sufficient, you do not need to derive the rules.

(c) Consider a process in which there are two incoming states with momenta p_1 and p_2 and three outgoing states with momenta k_i, $i = 1, 2, 3$. Starting from the expression of the S-matrix in terms of creation and annihilation operators, relate the S-matrix for this process to a correlation function. In your derivation you may use the LSZ approach or Dyson's formula.

(d) Draw all connected tree diagrams that contribute to the process in part (c). For diagrams that differ only by permutations of the external particles, it is sufficient to give a single diagram as part of your answer, provided you also indicate clearly the possible permutations.

(e) Based on the diagrams in part (d), and using Feynman rules, evaluate the contribution to the S-matrix corresponding to each connected tree diagram. For diagrams that differ only by permutations of the external particles, it is sufficient to find the expression for a single diagram, provided you also indicate clearly the possible permutations.

3

(a) Consider scalar QED (SQED), a theory consisting of a complex scalar field φ of mass M and an electromagnetic gauge field A_μ, with the only interaction being due to minimal coupling between them, with coupling constant e. Write down the Lagrangian $\mathcal{L}_{\text{SQED}}$ for this theory. Similarly, write down the Lagrangian $\mathcal{L}_{\text{QED}}$ for a Dirac spinor field ψ of mass m minimally coupled to a gauge field A_μ. Give the form of the gauge transformations under which $\mathcal{L}_{\text{SQED}}$ and $\mathcal{L}_{\text{QED}}$ are invariant. For each theory, derive the Noether current j_μ associated to the $U(1)$ global symmetry obtained by taking the gauge parameter to be constant.

(b) The charge conjugation operator $\hat{C}$ is defined to act on a complex scalar field operator by

$$\hat{C}^{-1}\varphi(x)\,\hat{C} = \varphi(x)^\dagger \ , \qquad \hat{C}^{-1}\varphi(x)^\dagger\,\hat{C} = \varphi(x) \ ,$$

where $\hat{C}$ is linear and $\hat{C}^\dagger\hat{C} = 1$. Show that $\mathcal{L}_{\text{SQED}}$ is invariant under charge conjugation provided A_μ transforms in a certain way, to be determined. You need not consider complications such as normal ordering arising from products of field operators.

(c) The charge conjugation operator $\hat{C}$ is defined to act on a Dirac spinor field operator by

$$\hat{C}^{-1}\psi(x)\,\hat{C} = \mathcal{C}\bar{\psi}(x)^T \ , \qquad \hat{C}^{-1}\bar{\psi}(x)\,\hat{C} = \psi(x)^T\mathcal{C}$$

where $\mathcal{C}$ is a 4×4 matrix that satisfies

$$\mathcal{C}^T = \mathcal{C}^{-1} = -\mathcal{C} \ , \qquad \mathcal{C}^{-1}(\gamma^\mu)^T\mathcal{C} = -\gamma^\mu \ .$$

Assuming that A_μ transforms as found in part (b), show that $\mathcal{L}_{\text{QED}}$ is invariant under charge conjugation. [*Hint:* $\bar{\psi}M\psi = -\psi^T M^T\bar{\psi}^T$*, for any matrix* M*.*] As in part (b), you need not consider complications such as normal ordering arising from products of field operators.

(d) Given that the vacuum state $|\Omega\rangle$ for the theories defined by $\mathcal{L}_{\text{SQED}}$ or $\mathcal{L}_{\text{QED}}$ is invariant under charge conjugation, show that

$$\langle\Omega|\mathrm{T}A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)|\Omega\rangle = 0 \ ,$$
$$\langle\Omega|\mathrm{T}j_{\mu_1}(x_1)\ldots j_{\mu_n}(x_n)|\Omega\rangle = 0 \ ,$$

provided that n is odd, where j_μ is the Noether current, as in part (a). You should prove that these results hold to all orders in the coupling constant, without making use of Feynman diagrams. What are the physical implications for scattering of photons in these theories?

4 In this question $u_L(x)$ and $u_R(x)$ will denote 2-component Weyl (or chiral) spinor fields that are left-handed and right-handed, respectively. We will also use the notation

$$\sigma^\mu = (1_{2\times 2}, \vec{\sigma}) \ , \qquad \bar{\sigma}^\mu = (1_{2\times 2}, -\vec{\sigma}) \ , \qquad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \ ,$$

where $1_{2\times 2}$ is the 2×2 identity matrix and σ^i $(i = 1, 2, 3)$ are the Pauli matrices. Standard properties of the Pauli matrices, e.g. $\sigma^i\sigma^j = \delta^{ij}1_{2\times 2} + i\epsilon^{ijk}\sigma^k$, can be used without proof.

Under a Lorentz boost with parameters $\vec{\chi} = (\chi^1, \chi^2, \chi^3)$, the Weyl spinors transform as

$$u_L \to e^{\vec{\chi}\cdot\vec{\sigma}/2} u_L \ , \qquad u_R \to e^{-\vec{\chi}\cdot\vec{\sigma}/2} u_R \ , \tag{1}$$

and under a rotation with parameters $\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$ they transform as

$$u_L \to e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_L \ , \qquad u_R \to e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_R \ . \tag{2}$$

(a) State clearly how the Dirac matrices γ^μ can be written in the Weyl representation in terms of σ^μ and $\bar{\sigma}^\mu$, so that a Dirac spinor can be decomposed in terms of Weyl spinors as

$$\psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \ .$$

Starting from the transformation properties of a Dirac spinor, which you should quote, verify that the transformation of u_L and u_R under a Lorentz boost is indeed given by (1), for a suitable choice of parameters.

Using (1) and (2), check explicitly that $u_L^\dagger u_R$ is invariant under the Lorentz group, and that $i\sigma^2 u_L^*$ transforms as a right-handed spinor under the Lorentz group.

(b) Consider the following Lagrangian for a left-handed spinor field of mass m:

$$\mathcal{L} = i u_L^\dagger \bar{\sigma}^\mu \partial_\mu u_L - i\frac{m}{2}\left(u_L^\dagger \sigma^2 u_L^* - u_L^T \sigma^2 u_L\right) \ , \tag{$\star$}$$

where all spinor components anticommute.

Show explicitly that this Lagrangian is Lorentz invariant, by considering transformations (1) and (2) with infinitesimal parameters χ^i and φ^i. You may use, without proof, that the required infinitesimal transformation of a vector V^μ is

$$\delta V^0 = -\chi^i V^i \ , \qquad \delta V^i = -\chi^i V^0 - \epsilon^{ijk}\varphi^j V^k \ .$$

From the Lagrangian ($\star$), derive the equations of motion for u_L.

Can this Lagrangian describe a massive spin-1/2 particle with non-zero electric charge? Give a brief justification for your answer.

(c) Write down the Dirac Lagrangian for a Dirac spinor ψ of mass m. Show that, for an appropriate form of ψ which you should specify, the Dirac Lagrangian reduces to ($\star$), up to a multiplicative constant and a total derivative.

Similarly, show that the Dirac equation for ψ of this specified form reduces to the equation of motion found in part (b).

END OF PAPER