UNIVERSITY OF CAMBRIDGE

MAMA/211, NST3AS/211, MAAS/211

MAT3
MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 9:00 am to 12:00 pm

PAPER 211

ADVANCED FINANCIAL MODELS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
Question 1 carries 40 marks. Questions 2, 3 and 4 each carry 20 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a discrete-time market model consisting of n assets with prices $P = (P^1, \ldots P_t^n)_{t \geqslant 0}$ and paying dividends $\delta = (\delta_t^1, \ldots, \delta_t^n)_{t \geqslant 1}$. For an n-dimensional previsible process H define the notation

$$\begin{aligned} \pi_t^H &= H_{t+1} \cdot P_t \text{ for all } t \geqslant 0 \\ \xi_t^H &= H_t \cdot (P_t + \delta_t) - \pi_t^H \text{ for all } t \geqslant 1. \end{aligned}$$

(a) What is an *arbitrage* in this market?

(b) What is a *martingale deflator* in this market? If Y is a martingale deflator, show that for fixed H, the process Z defined by

$$Z_t = \pi_t^H Y_t + \sum_{s=1}^{t} \xi_s^H Y_s$$

is a local martingale. [Properties of the martingale transform, as proven in lectures, can be used without proof.]

(c) State the *first fundamental theorem of asset pricing.*

From now on, assume that this market has no arbitrage.

(d) For a fixed H, prove that the market with $n+1$ assets with prices $\tilde{P} = (P^1, \ldots, P^n, \pi^H)$ and dividends $\tilde{\delta} = (\delta^1, \ldots, \delta^n, \xi^H)$ also has no arbitrage.

(e) Let η be a previsible process such that π^η is previsible with $\pi_0^\eta > 0$, and $\xi_t^\eta = 0$ for all $t \geqslant 1$. Show that $\pi_t^\eta > 0$ for all $t > 0$.

(f) Let K be such that $\pi_t^K = 1$ for all $t \geqslant 0$ and that ξ^K is previsible. Show that

$$\xi_t^K = \frac{\pi_t^\eta}{\pi_{t-1}^\eta} - 1 \text{ for all } t \geqslant 1$$

where η is the process from part (e).

(g) Show that the following are equivalent:

(1) For every non-random $T > 0$ and $\mathcal{F}_T$ measurable random variable X_T, there exists a $\mathcal{F}_{T-1}$-measurable random vector H_T such that $X_T = H_T \cdot (P_T + \delta_T)$.

(2) For every non-random $T > 0$ and every adapted process $(X_t)_{1 \leqslant t \leqslant T}$ there exists a previsible process H such that $H_t = 0$ for $t \geqslant T + 1$ and $\xi_t^H = X_t$ for all $1 \leqslant t \leqslant T$.

(h) Suppose one of the equivalent conditions from part (g) holds. Show that there can exist at most one martingale deflator Y such that $Y_0 = 1$.

2 Consider a T-period market with prices $P = (P_t)_{0 \leqslant t \leqslant T-1}$ and dividends $\delta = (\delta_t)_{1 \leqslant t \leqslant T}$. Suppose that one of the assets is a (zero-coupon) bond of maturity T with time-t price B_t^T and time-T dividend of 1. Assume that $B_t^T > 0$ for all $0 \leqslant t \leqslant T-1$.

(a) What is a T-*forward measure* for this model?

For the rest of the question, let $\mathbb{Q}^T$ be a T-forward measure. Furthermore, suppose that the market contains a stock with non-negative time-T payout of S_T. Suppose also that there is a family of forward contracts on the stock, each of maturity T, initiated at times $t \in \{0, \ldots, T-1\}$. Assume that the forward contract initiated at time t has $\mathcal{F}_t$-measurable strike F_t^T, and that the price of the forward contract at initiation is zero. That is, assume that F_t^T is the time-t forward price of the stock for maturity T.

(b) Show that $(F_t^T)_{0 \leqslant t \leqslant T-1}$ is a martingale under $\mathbb{Q}^T$.

Suppose that the market has a family of European call options on the stock, each with maturity T, with strikes in a finite set $\mathcal{K} \subset \mathbb{R}_+$. Let the time-$t$ price of the call option with strike K be $C_t^{T,K}$ for $0 \leqslant t \leqslant T-1$.

(c) Show that $K \mapsto C_t^{T,K}$ is non-increasing for all $0 \leqslant t \leqslant T-1$.

Suppose that the market contains a European contingent claim on the stock with time-T payout $g(S_T)$, where g is a smooth convex function. Let π_t be the time-t price of the claim.

(d) Show that

$$\pi_t \geqslant B_t^T \left(g(0) + g'(0) F_t^T\right) + \sum_{i=1}^{N} \left(g'(K_i) - g'(K_{i-1})\right) C_t^{T,K_i}$$

for all $0 \leqslant t \leqslant T-1$, where $\mathcal{K} = \{K_1, \ldots, K_N\}$ and $K_0 = 0$.

3 Throughout this question you may use standard results on stochastic calculus if carefully stated. Consider a continuous-time market with three assets. The first is cash with constant price $B_t = 1$ for all $t \geqslant 0$. The second is a stock with price $S = (S_t)_{t \geqslant 0}$ whose dynamics are given by

$$\begin{aligned} dS_t &= S_t \sqrt{v_t}(\rho dW_t + \sqrt{1-\rho^2} dW_t^\perp) \\ dv_t &= (\alpha - \beta v_t)dt + \gamma \sqrt{v_t} dW_t \end{aligned}$$

where $S_0, v_0, \alpha, \beta, \gamma$ are positive constants and $-1 \leqslant \rho \leqslant 1$, and where W and $W^\perp$ are independent Brownian motions generating the filtration $(\mathcal{F}_t)_{t \geqslant 0}$. The third asset is a European claim on the stock with time-T payout of $g(S_T)$ and time-t price of $\pi_t = U(t, v_t, S_t)$ where $U : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a bounded function satisfying

$$\begin{aligned} &U(T,v,s) = g(s) \text{ for all } v \geqslant 0, s \geqslant 0, \\ &\frac{\partial U}{\partial t} + (\alpha - \beta v)\frac{\partial U}{\partial v} + \frac{v}{2}\left(\gamma^2 \frac{\partial^2 U}{\partial v^2} + 2s\gamma\rho \frac{\partial^2 U}{\partial v \partial s} + s^2 \frac{\partial^2 U}{\partial s^2}\right) = 0. \end{aligned}$$

(a) Why is there no arbitrage in this market relative to cash? [Results on arbitrage-theory as proven in lectures may be used if carefully stated.]

(b) Show that $\pi_0 = \mathbb{E}[g(S_T)]$.

Let $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$G(c,q) = \int_{-\infty}^{\infty} g(ce^{-\frac{1}{2}q + \sqrt{q}z}) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

(c) Show that

$$\pi_0 = \mathbb{E}\left[G\left(\tilde{S}_T, (1-\rho^2)Y_T\right)\right]$$

where

$$\begin{aligned} d\tilde{S}_t &= \tilde{S}_t \rho \sqrt{v_t} dW_t, & \tilde{S}_0 &= S_0 \\ dY_t &= v_t \, dt, & Y_0 &= 0 \end{aligned}$$

(d) Show that

$$\tilde{S}_T = S_0 \exp\left(\tfrac{\rho}{\gamma}(v_T - v_0 - \alpha T + \beta Y_T) - \tfrac{1}{2}\rho^2 Y_T\right).$$

Let $\tilde{U} : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a bounded solution of

$$\begin{aligned} &\tilde{U}(T,v,y) = G(S_0 e^{\frac{\rho}{\gamma}(v - v_0 - \alpha T) + (\frac{\rho\beta}{\gamma} - \frac{1}{2}\rho^2)y}, (1-\rho^2)y), \text{ for all } v \geqslant 0, y \geqslant 0 \\ &\frac{\partial \tilde{U}}{\partial t} + (\alpha - \beta v)\frac{\partial \tilde{U}}{\partial v} + \frac{v}{2}\left(\gamma^2 \frac{\partial^2 \tilde{U}}{\partial v^2} + 2\frac{\partial \tilde{U}}{\partial y}\right) = 0 \end{aligned}$$

(e) Show that $\pi_0 = \tilde{U}(0, v_0, 0)$.

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(a) Let U be a discrete-time supermartingale. Show that there exists a martingale M and a previsible non-decreasing process A such that $M_0 = 0 = A_0$ and $U_t = U_0 + M_t - A_t$ for all $t \geqslant 0$.

Let $Z = (Z_t)_{0 \leqslant t \leqslant T}$ be a discrete-time integrable adapted process. Let U be defined by

$$\begin{aligned} U_T &= Z_T \\ U_t &= \max\{Z_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} \text{ for all } 0 \leqslant t \leqslant T-1. \end{aligned}$$

All stopping times in this problem take values in $\{0, \dots, T\}$.

(b) Let τ be a stopping time. Show that $U_0 \geqslant \mathbb{E}(Z_\tau)$.

(c) Find a stopping time τ^* such that $U_0 = \mathbb{E}(Z_{\tau^*})$.

(d) Let X be a martingale such that $X_0 = 0$. Show that

$$U_0 \leqslant \mathbb{E}(\max_{0 \leqslant t \leqslant T}\{Z_t + X_t\}).$$

[Hint: First explain why $\mathbb{E}(Z_\tau + X_\tau) = \mathbb{E}(Z_\tau)$ for any stopping time τ.]

(e) Show that there exists a martingale X^* with $X_0 = 0$ such that

$$U_0 = \mathbb{E}(\max_{0 \leqslant t \leqslant T}\{Z_t + X_t^*\}).$$

END OF PAPER