UNIVERSITY OF CAMBRIDGE

MAMA/209, NST3AS/209, MAAS/209

MAT3
MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2024 9:00 am to 11:00 am

PAPER 209

LATTICE MODELS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Harris' inequality for percolation

Let $p_1, \ldots, p_n$ be fixed in $(0, 1)$. Let $X_1, \ldots, X_n$ be n independent random variables such that $P[X_i = 1] = 1 - P[X_i = 0] = p_i$ for all $i \in \{1, \ldots, n\}$.

Let A and B be two increasing events in $\{0, 1\}^n$. How do $P[(X_1, \ldots, X_n) \in A \cap B]$ and $P[(X_1, \ldots, X_n) \in A] \times P[(X_1, \ldots, X_n) \in B]$ compare? Provide a proof [for instance using the coupling between a well-chosen Markov chain on $\{0, 1\}^n$ and a well-chosen Markov chain on A].

2 How to determine p_c for site percolation on the triangular lattice

1) Explain how one proves that $1/2 \geqslant p_c$ for site percolation on the triangular lattice, using the following two ingredients (you do not need to reprove these ingredients here – the question is how to combine them to deduce that $1/2 \geqslant p_c$):

- Exponential decay of connection probabilities in subcritical percolation.
- The left-to-right crossing probability of an $N \times N$ rhombus is $1/2$ for site percolation on the triangular lattice at $p = 1/2$.

2) Explain how to show that there is no infinite cluster at $p = 1/2$ for site percolation on the triangular lattice using the Russo-Seymour-Welsh estimates (again, you do not need to prove the Russo-Seymour-Welsh estimates, just how to use them to show that there is almost surely no infinite cluster).

3 Gaussian Free Field in the discrete torus

In this question, the answers can be inspired by results and proofs that were presented in the lectures, but they are expected to be self-contained.

Let $N \geqslant 2$. We consider the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$. We call S_N the set of all N^d sites in this torus, and E_N the set of all dN^d edges joining neighbouring sites. In this graph T_N with vertex-set S_N and edge-set E_N, each site has exactly $2d$ neighbours.

For each real-valued function F defined on S_N, we define $\mathcal{E}(F)$ to be the sum over all edges e on T_N of $(F(y_e) - F(x_e))^2$ (where x_e and y_e are the end-points of the edge e).

For each $a \in S_N$, we define the Gaussian Free Field in $S_N^a := S_N \setminus \{a\}$ with Dirichlet condition at a to be a random function $(\Gamma(x))_{x \in S_N}$ such that $\Gamma(a) = 0$ almost surely and such that the law of $(\Gamma(x), x \in S_N^a)$ has a density on $\mathbb{R}^{S_N^a}$ that is proportional to $\exp(-\mathcal{E}(\gamma)/(2d))$ at all $(\gamma(x), x \in S_N^a)$ (where by convention, $\gamma(a) = 0$ when one then defines $\mathcal{E}(\gamma)$).

1) We now fix another point b in S_N. Show that $(\Gamma(x) - \Gamma(b))_{x \in S_N}$ is then a GFF in S_N^b with Dirichlet boundary condition at b. [Hint: Just look at the density function of the obtained process, and not at its covariance structure].

2) Let $x_0 \in S_N^a$. What is the conditional distribution of $\Gamma(x_0)$ given $(\Gamma(x), x \neq x_0)$?

3) For each non-empty subset R of S_N, we define $G_R(x,y)$ to be the expected number of times a random walk in T_N started at x does visit y before hitting R. What is the relation between $E[\Gamma(x)\Gamma(y)]$ and $G_{\{a\}}(x,y)$. Justify this answer.

4) Let $x_1, \ldots, x_K$ be an ordering of $S_N \setminus \{a\}$. Show that the product

$$G_{\{a\}}(x_1, x_1) G_{\{a, x_1\}}(x_2, x_2) \ldots G_{\{a, x_1, \ldots, x_{K-1}\}}(x_K, x_K)$$

is in fact independent of the chosen ordering. [Hint: Use the spatial Markov property of the GFF and look at the density of the law of Γ at $(0, \ldots, 0)$].

4 Loop-erased random walks in the discrete torus

[In this question, the answers can be inspired by results and proofs that were presented in the lectures, but they are expected to be self-contained. One will however be allowed to use without proof the last statement of Question 3].

Let $N \geqslant 2$. We consider the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$. We call S_N the set of all N^d sites in this torus, and E_N the set of all dN^d edges joining neighbouring sites. In this graph $T_N = (S_N, E_N)$, each site has exactly $2d$ neighbours.

Fix two vertices x_1 and a in T_N and consider the random simple path L from x_1 to a obtained by sampling a random walk started from x_1 and stopped at the first time at which it hits a, and then loop-erasing this stopped random walk path chronologically. We call the obtained path $(L(0) = x_1, L(1), \dots, L(\tau) = a)$ a loop-erased walk from x_1 to a in T_N.

We say that a random collection $\mathcal{T}$ of edges in E_N is a uniform spanning tree in T_N if it is chosen uniformly among the subsets F of E_N such that (S_N, F) is a graph with one connected component and no cycle.

1) Explain why the unique simple path with edges in $\mathcal{T}$ that connects x_1 to a is distributed like a loop-erased random walk from x_1 to a in T_N. [Hint: Consider Wilson's algorithm thinking of a as the boundary point, and use the property stated at the end of Question 3 to see that it constructs a uniform spanning tree].

2) Deduce that if $(L(0) = x_1, L(1), \dots, L(\tau) = a)$ is loop-erased random walk from x_1 to a in T_N, then $(L(\tau), L(\tau - 1), \dots, L(1), L(0))$ has the same law of a loop-erased walk from a to x_1 in T_N.

3) Can you deduce a similar result for loop-erased random walks in $\mathbb{Z}^2$ instead of T_N?

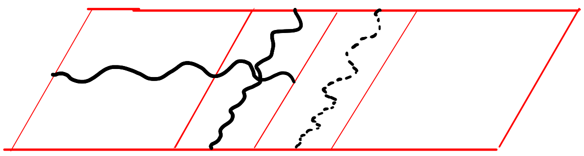
5 We consider percolation of parameter p for site percolation on the triangular lattice and call P_p the corresponding probability measure on the space of percolation configurations. We define $P_p(E_N)$ to be the probability of the event E_N that there exists a left-to-right crossing of the $6N \times 2N$ parallelogram $W_n := \{n + \tau m \ : \ -3N \leqslant n \leqslant 3N, -N \leqslant n \leqslant N\}$ (with $\tau = \exp(i\pi/3)$ and viewing the plane as the set of complex numbers).

1) How can one describe a pivotal point for the particular increasing event E_N and a percolation realization $w : W_n \to \{0, 1\}$? Recall (without proof) the result relating the expected number of such pivotal points to $dP_p(E_N)/dp$.

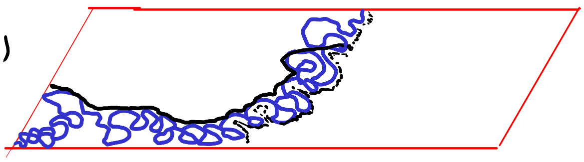
2) Outline a proof of the fact for all $N = 3^j \geqslant 9$, $dP_p(E_N)/dp$ at $p = 1/2$ is bounded from below by some positive constant times j.

Hint: One can try to repeatedly use the Russo-Seymour-Welsh estimates and exploration processes discovering interfaces between closed and open clusters, and to be inspired by the following sequences of sketchy figures [for instance, one can start noting that when the event with two open and one closed crossings sketched in (i) holds, then the exploration process started from the bottom left corner hits the top boundary as in (ii)].

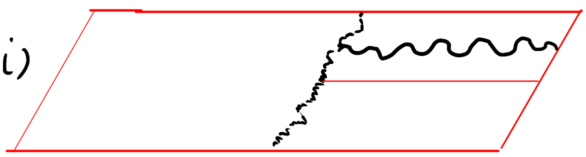
(i)



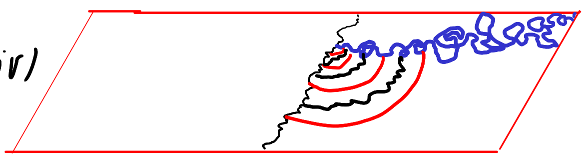
(ii)



(iii)



(iv)



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