UNIVERSITY OF CAMBRIDGE

MAMA/203, NST3AS/203, MAAS/203

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 11 June 2024 1:30 pm to 3:30 pm

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For a compact $\mathbb{H}$-hull $A \subseteq \mathbb{H}$, define its *mapping-out function* g_A and its *half-plane capacity* $\operatorname{hcap}(A)$. Show that $\operatorname{hcap}(\lambda A + x) = \lambda^2 \operatorname{hcap}(A)$ for $\lambda > 0$ and $x \in \mathbb{R}$.

(b) (i) Show that there exists $c > 0$ such that $\operatorname{hcap}(A) \leqslant c \operatorname{diam}(A)^2$ for every compact $\mathbb{H}$-hull A.

(ii) Find a sequence of compact $\mathbb{H}$-hulls such that $\operatorname{hcap}(A_n) \to 0$ but $\operatorname{diam}(A_n) \nrightarrow 0$. *[You may use that as $y \to \infty$, the probability that a two-dimensional Brownian motion starting at iy exits $\mathbb{H} \setminus \mathbb{D}$ on $\partial\mathbb{D}$ is comparable to y^{-1}.]*

(c) Let $\theta \in]0, \pi[$, and let a be the half-plane capacity of the straight line segment $\{re^{i\theta} \mid r \in]0, 1]\}$. Find the half-plane capacity parametrisation of the ray $\{re^{i\theta} \mid r > 0\}$. That is, find $r(t)$ such that if $\gamma(t) = r(t)e^{i\theta}$, then $\operatorname{hcap}(\gamma([0, t])) = 2t$ for $t \geqslant 0$. *[You do not need to specify the exact value of a.]*

(d) Let (K_t) be the family of hulls generated by γ, i.e. $K_t = \gamma([0, t])$. Show that there exists $c \in \mathbb{R}$ such that the driving function of (K_t) is $U_t = c\sqrt{t}$. *[You may assume that (K_t) has the local growth property.]*

2

(a) State and prove the *scaling invariance* and the *conformal Markov property* of SLE_κ in $(\mathbb{H}, 0, \infty)$. *[You may use any deterministic properties of Loewner chains.]*

(b) Let $(g_t)_{t \geqslant 0}$ be the SLE_κ Loewner chain driven by $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion. For $T > 0$, let $(h_s)_{s \in [0,T]}$ be defined by

$$\partial_s h_s(z) = \frac{-2}{h_s(z) - (U_{T-s} - U_T)}, \quad h_0(z) = z$$

where $z \in \mathbb{H}$. Show that $h_T(z) = g_T^{-1}(U_T + z) - U_T$. Is it true that $h_s(z) = g_s^{-1}(U_s + z) - U_s$ for $s \in]0, T[$?

(c) Suppose that $\kappa > 8$. Show that there exists a deterministic $\alpha > 0$ and a random variable C such that $|h_T'(x + iy)| \leqslant C y^{-1+\alpha}$ for all $x \in [-1, 1]$, $y \in]0, 1]$. *[You may use that if $z \in \mathbb{H}$ and $X_s + iY_s = h_s(z) - (U_{T-s} - U_T)$, then $M_s = |h_s'(z)|^2 (1 + X_s^2/Y_s^2)^{4/\kappa}$ is a martingale for $s \in [0, T]$.]*

(d) Deduce (for $\kappa > 8$) that the map h_T is almost surely Hölder-continuous on $[-1, 1] \times]0, 1]$.

3 Let $1 < d < 2$, and B be a standard Brownian motion. Suppose

$$x > 0 > y$$

and we have two processes X, Y satisfying

$$X_t = x + B_t + \frac{d-1}{2} \int_0^t \frac{1}{X_s} \, ds, \qquad t < \tau_x = \inf\{s > 0 \mid X_s = 0\},$$

$$Y_t = y + B_t + \frac{d-1}{2} \int_0^t \frac{1}{Y_s} \, ds, \qquad t < \tau_y = \inf\{s > 0 \mid Y_s = 0\}.$$

The goal of this question will be to compute the probability $\mathbb{P}[\tau_x > \tau_y]$. *[You may assume that $\tau_x < \infty$ and $\tau_y < \infty$ almost surely.]*

(a) Explain what the probability $\mathbb{P}[\tau_x > \tau_y]$ means in term of hitting events of an SLE_κ trace.

(b) Explain why $\mathbb{P}[\tau_x > \tau_y]$ depends only on the ratio x/y.

(c) For $v = x/y < 0$, define $F(v) = F(x/y) = \mathbb{P}[\tau_x > \tau_y]$. Consider the process $V_t = X_t/Y_t$ for $t < \tau_x \wedge \tau_y$. You may use that

$$dV_t = (1 - V_t)\frac{dB_t}{Y_t} + (\frac{d-1}{2}\frac{1}{V_t} + \frac{3-d}{2}V_t - 1)\frac{dt}{Y_t^2}.$$

Assuming that the function F is C^2, show that it satisfies

$$F''(v) + \left(\frac{d-1}{v} + \frac{2d-4}{1-v}\right) F'(v) = 0 \quad \text{for } v < 0. \tag{1}$$

[You may use that $\frac{d-1}{v(1-v)^2} + \frac{(3-d)v}{(1-v)^2} - \frac{2}{(1-v)^2} = \frac{d-1}{v} + \frac{2d-4}{1-v}$.]

(d) We now define

$$\widetilde{F}(v) = \int_v^0 \frac{du}{(-u)^{d-1}(1-u)^{4-2d}}$$

and note that it is C^2 and satisfies (1). *[You do not need to verify this.]* What can you say about the process $\widetilde{F}(V_t)$ for $t < \tau_x \wedge \tau_y$? Conclude that $F(v) = c\widetilde{F}(v)$ for some constant $c > 0$.

4 Let $\mathcal{M}$ denote the set of finite Borel measures ρ on the unit disc $\mathbb{D}$ with $\iint_{\mathbb{D}} G_{\mathbb{D}}(x,y)\rho(dx)\rho(dy) < \infty$ where $G_{\mathbb{D}}$ is the zero-boundary Green's function on $\mathbb{D}$.

(a) Define the *zero-boundary Gaussian free field* on $\mathbb{D}$ as a stochastic process $((h,\rho))_{\rho\in\mathcal{M}}$. State the *domain Markov property* of the Gaussian free field.

(b) For $t \geqslant 0$, let ρ_t be the uniform measure on $\partial B(0,e^{-t})$, and define the process $X_t = (h,\rho_t)$. *[You do not need to check that $\rho_t \in \mathcal{M}$. You may further use that the process (X_t) has a version that is continuous.]*
Now let

$$h = h_{\mathbb{D}\setminus B(0,e^{-t})} + h_{B(0,e^{-t})}^{\mathbb{D}\setminus B(0,e^{-t})}$$

be as in the domain Markov decomposition of h. Explain why $(h_{\mathbb{D}\setminus B(0,e^{-t})}, \rho_s) = h_{\mathbb{D}\setminus B(0,e^{-t})}(0)$ for every $s > t$.

(c) Show that for $s > t$, the increment $X_s - X_t$ is independent of $h_{\mathbb{D}\setminus B(0,e^{-t})}$ and has the same law as X_{s-t}.

(d) Deduce that the process (X_t) has independent and Gaussian stationary increments. Conclude that its law is that of a multiple of a standard Brownian motion.

END OF PAPER