UNIVERSITY OF CAMBRIDGE

MAMA/202, NST3AS/202, MAAS/202

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 9:00 am to 12:00 pm

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let β be a standard Brownian motion and let

$$B_t = \int_0^t \operatorname{sign}(\beta_s) d\beta_s.$$

(i) Show that B is a standard Brownian motion.

(ii) Show that B_t and β_t are uncorrelated for each $t > 0$.

(iii) Prove or disprove: B_t and β_t are independent for each $t > 0$. *[Hint: consider β_t^2.]*

(b) (i) State the Dubins-Schwarz theorem.

(ii) Let B be a standard Brownian motion and let H be a locally bounded previsible process such that $\int_0^\infty H_s^2 ds = \infty$ a.s. Let $\tau = \inf\{t \geqslant 0 : \int_0^t H_u^2 du = 1\}$. Identify the distribution of $\int_0^\tau H_u dB_u$.

(iii) Suppose that $M \in \mathcal{M}_{c,loc}$ and let $B_s = M_{\tau_s}$ where $\tau_s = \inf\{t \geqslant 0 : [M]_t > s\}$. Prove or disprove: B is independent of $[M]$.

(c) Suppose that B^1, B^2 are independent standard Brownian motions. Express the distribution of $\int_0^t B_s^1 dB_s^2 + \int_0^t B_s^2 dB_s^1$ in terms of $N(0,1)$ random variables.

2

(a) Explain what it means for a cadlag process to be of *finite variation* and give the definition of *ucp convergence.*

(b) Suppose that $M, N \in \mathcal{M}_{c,loc}$ and let V be the total variation process for $[M, N]$.

(i) For each $t \geqslant 0$, let

$$\widetilde{V}_t^n = \sum_{k=0}^{\lceil 2^n t \rceil - 1} |M_{(k+1)2^{-n}} - M_{k2^{-n}}||N_{(k+1)2^{-n}} - N_{k2^{-n}}|.$$

Prove that there exists a continuous process $\widetilde{V}$ so that $\widetilde{V}^n \to \widetilde{V}$ ucp as $n \to \infty$.

(ii) Prove that $V_t \leqslant \widetilde{V}_t$ for all $t \geqslant 0$.

(iii) Prove that $V_t \leqslant [M]_t^{1/2}[N]_t^{1/2}$ for all $t \geqslant 0$.

(c) Suppose that $M \in \mathcal{M}_c^2$ and $H \in L^2(M)$. Suppose that $N \in \mathcal{M}_c^2$ and assume that

$$[N, K]_t = \int_0^t H_s d[M, K]_s \quad \text{for all} \quad K \in \mathcal{M}_c^2.$$

(i) Show that $\mathbf{E}[N_\infty K_\infty] = \mathbf{E}[[N, K]_\infty]$ and $\mathbf{E}[(H \cdot M)_\infty K_\infty] = \mathbf{E}[[H \cdot M, K]_\infty]$ for all $K \in \mathcal{M}_c^2$.

(ii) Show that $N = H \cdot M$.

[You may use results from lectures provided you state them clearly.]

3

(a) Suppose that $M \in \mathcal{M}_c$ with $M_0 = 0$ and assume further that M is a Gaussian process.

(i) Prove that for every $s, t > 0$ we have that $M_{t+s} - M_t$ is independent of $\mathcal{F}_t = \sigma(M_r : 0 \leqslant r \leqslant t)$.

(ii) Show that there exists a (deterministic) continuous function $f\colon [0,\infty) \to [0,\infty)$ so that $[M]_t = f(t)$ for all $t \geqslant 0$.

(b) Prove or disprove: if X is a continuous Gaussian process with $\mathbf{E}[X_t] = 0$ for all t then $X \in \mathcal{M}_c$.

(c) Prove or disprove: if $M \in \mathcal{M}_c$ and there exists a deterministic continuous function so that $[M]_t = f(t)$ for all $t \geqslant 0$ then M is a Gaussian process.

(d) (i) Suppose that B is a standard Brownian motion in $\mathbf{R}^2$ with $|B_0| = r \in (0,\infty)$. Fix $0 < r_1 < r < r_2$ and let $\tau = \inf\{t \geqslant 0 : |B_t| \notin (r_1, r_2)\}$. Prove that

$$\mathbf{P}[|B_\tau| = r_1] = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}.$$

[You may use without proof that $\Delta \log |z| = 0$ for all $z \in \mathbf{R}^2 \setminus \{0\}$.]

(ii) Prove or disprove: there exists a continuous function u on $\{(x,y) \in \mathbf{R}^2 : |(x,y)| \leqslant 1\}$ such that

$$\begin{cases} \Delta u = 0 & \text{on} \quad \{(x,y) \in \mathbf{R}^2 : 0 < |(x,y)| < 1\} \\ u = 0 & \text{on} \quad \{(x,y) \in \mathbf{R}^2 : |(x,y)| = 1\}, \quad \text{and} \\ u((0,0)) = 1. \end{cases}$$

4 Fix $\delta \in \mathbf{R}$. Recall that the Bessel SDE is given by

$$dX_t = \frac{\delta - 1}{2X_t} dt + dB_t, \quad X_0 = x. \tag{1}$$

Assume that $\delta \geqslant 1$ and B is a standard Brownian motion.

(a) Let $Y_t = |B_t|^\delta$. Suppose that τ is a finite stopping time for B so that $B_\tau \neq 0$ a.s. and $\sigma = \inf\{t \geqslant \tau : B_t = 0\}$. Explain why

$$dY_t = \delta Y_t^{(\delta-1)/\delta} dB_t + \frac{\delta(\delta-1)}{2} Y_t^{(\delta-2)/\delta} dt \quad \text{for} \quad t \in [\tau, \sigma].$$

(b) For each $s \geqslant 0$, let $\tau_s = \inf\{t \geqslant 0 : \int_0^t \delta^2 Y_u^{2(\delta-1)/\delta} du = s\}$. Explain why τ_s is strictly increasing and $\tau_s < \infty$ a.s. for each $s \geqslant 0$.

(c) Let $X_s = Y_{\tau_s}$ and show that X_s is continuous.

(d) Let τ be a finite stopping time for X so that $X_\tau \neq 0$ a.s. and let $\sigma = \inf\{t \geqslant \tau : X_t = 0\}$. Show that there exists a standard Brownian motion $\widetilde{B}$ so that

$$dX_t = \frac{\delta - 1}{2X_t} dt + d\widetilde{B}_t \quad \text{for} \quad t \in [\tau, \sigma].$$

(e) Compare the solution to (1) constructed in the previous parts to the solution constructed from the maximal local existence theorem from the course.

(f) Show that $\{t \geqslant 0 : X_t = 0\}$ has zero Lebesgue measure a.s.

5

(a) (i) Suppose that $\mathbf{P}$, $\widetilde{\mathbf{P}}$ are probability measures. Explain what it means for $\mathbf{P}$ to be absolutely continuous with respect to $\widetilde{\mathbf{P}}$.

(ii) Define the stochastic exponential $Z = \mathcal{E}(M)$ of $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$.

(iii) State Girsanov's theorem.

(b) Suppose that $0 < T < \infty$ and $b \colon \mathbf{R} \to \mathbf{R}$ is a bounded, continuous function. Prove that the SDE

$$dX_t = b(X_t)dt + dB_t \quad \text{for} \quad t \in [0, T]$$

has a weak solution and satisfies uniqueness in law.

(c) Suppose that B is a standard Brownian. In each of the following, prove or disprove that the law of $\widetilde{B}|_{[0,1]}$ is absolutely continuous with respect to the law of $B|_{[0,1]}$.

(i) $\widetilde{B}_t = B_{2t}$ for each $t \geqslant 0$.

(ii) Suppose that $f \in C^1$ with $f(0) = 0$ and $\widetilde{B}_t = B_t + f(t)$ for each $t \geqslant 0$.

(iii) Suppose that $f \colon [0, \infty) \to \mathbf{R}$ is continuous with $f(0) = 0$ and $\widetilde{B}_t = B_t + f(t)$ for each $t \geqslant 0$.

[You may use without proof the precise Hölder regularity of B provided you state it clearly.]

6 Let B be a standard Brownian motion and let $\mathbf{E}_x$ denote the expectation with respect to the probability measure under which $B_0 = x$.

(a) Suppose that $f\colon [0,\infty) \to \mathbf{R}$ is a bounded continuous function and that $u \in C^{1,2}((0,\infty)\times(0,\infty)) \cap C([0,\infty)\times[0,\infty))$ and for some $C \in (0,\infty)$ (deterministic) satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{on} \quad (0,\infty)\times(0,\infty), \\ u(0,x) = f(x) & \text{for} \quad x \in (0,\infty), \\ \lim_{x\to 0^+} \frac{\partial u}{\partial x}(t,x) = 0 & \text{for} \quad t \in (0,\infty), \\ |u(t,x)| \leqslant Ce^{Cx} & \text{for} \quad (t,x) \in (0,\infty)\times(0,\infty). \end{cases} \tag{1}$$

Show that

$$u(t,x) = \mathbf{E}_x[f(|B_t|)] \quad \text{for} \quad (t,x) \in (0,\infty)\times(0,\infty).$$

[You may assume without proof that the function $(0,\infty)\times\mathbf{R} \to \mathbf{R}$ *defined by* $(s,x) \mapsto u(s,|x|)$ *is in* $C^{1,2}((0,\infty)\times\mathbf{R})$*.]*

(b) Suppose that we have the setting of part (a) and we set $v(t,x) = \mathbf{E}_x[f(|B_t|)]$. Show that $v \in C^{1,2}((0,\infty)\times(0,\infty)) \cap C([0,\infty)\times[0,\infty))$ and satisfies (1).

(c) Suppose that $f_1, f_2\colon (0,\infty) \to \mathbf{R}$ and $b\colon (0,1) \to \mathbf{R}$ are bounded continuous functions and suppose that $u \in C^{1,2}((0,\infty)\times(0,1)) \cap C([0,\infty)\times[0,1])$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{on} \quad (0,\infty)\times(0,1), \\ u(0,x) = g(x) & \text{for} \quad x \in (0,1), \\ u(t,0) = f_1(t) & \text{for} \quad t \in (0,\infty), \\ u(t,1) = f_2(t) & \text{for} \quad t \in (0,\infty). \end{cases}$$

For each $r \in \mathbf{R}$ let $\tau_r = \inf\{t \geqslant 0 : B_t = r\}$. Show that

$$u(t,x) = \mathbf{E}_x[g(B_t)\mathbf{1}_{\{t<\tau_0\wedge\tau_1\}} + f_1(t-\tau_0)\mathbf{1}_{\{\tau_0<t\wedge\tau_1\}} + f_2(t-\tau_1)\mathbf{1}_{\{\tau_1<t\wedge\tau_0\}}].$$

END OF PAPER