UNIVERSITY OF CAMBRIDGE

MAMA/201, NST3AS/201, MAAS/201

MAT3
MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 9:00 am to 12:00 pm

PAPER 201

ADVANCED PROBABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(X_i)_i$ be an i.i.d. sequence of random variables with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = -1) = 1/2$. Let $S_0 = 0$, and for $n \geqslant 1$, define

$$S_n = \sum_{i=1}^{n} X_i.$$

(a) Show that if T is a stopping time of finite mean, then

$$\mathbb{E}\left[S_T^2\right] = \mathbb{E}[T]\,.$$

[*Hint: You may wish to use the* $\mathcal{L}^2$ *convergence theorem for* $(S_{n\wedge T})_{n\geqslant 0}$.]

Is this equality of means true when T has infinite mean? Justify your answer.

(b) For $c > 0$, we define $T_c = \inf\{n \geqslant 0 : |S_n|^2 > c^2 n\}$. Show that T_c is finite almost surely.

(c) For $c > 1$, find $\mathbb{E}[T_c]$.

2 (a) Define the notions of *weak convergence* for probability measures and random variables.

(b) A sequence of probability measures (μ_n) in $\mathbb{R}$ is said to converge to the probability measure μ in total variation if

$$\sup_{A \in \mathcal{B}(\mathbb{R})} |\mu_n(A) - \mu(A)| \to 0 \quad \text{as } n \to \infty.$$

Show that total variation convergence implies weak convergence. Does the converse hold? Justify your answer.

(c) Let $(X_n)_{n\in\mathbb{N}}$ and X be random variables satisfying for some $C > 0$,

$$\forall\, n \quad |X_n| \leqslant C \quad \text{and} \quad |X| \leqslant C.$$

Show that the following two statements are equivalent:

(i) X_n converges weakly to X, and

(ii) $\mathbb{E}\big[X_n^k\big] \to \mathbb{E}\big[X^k\big]$ as $n \to \infty$ for all $k \in \mathbb{N}$.

[*Hint: You may use without proof the Weierstrass approximation theorem: given f a continuous function on $[a,b]$, for every $\epsilon > 0$, there exists a polynomial P such that* $\sup_{x\in[a,b]} |f(x) - P(x)| \leqslant \epsilon$.]

(d) Let M, M' be two metric spaces, and let $f : M \to M'$ be a measurable function. Suppose that (X_n) and X are random variables with values in M. Let D_f be the set of discontinuity points of f. Show that if $\mathbb{P}(X \in D_f) = 0$ and (X_n) converges weakly to X, then $(f(X_n))$ also converges weakly to $f(X)$ as $n \to \infty$.

[Throughout this question you may use results from the course provided they are stated clearly.]

3 Let (B_t) be a two dimensional Brownian motion. For $a > 0$, define

$$T_a = \inf\{t \geqslant 0 : |B_t| = a\}.$$

(a) Suppose that $B_0 = x$ and $0 < r < R$ satisfy $r < |x| < R$. Show that

$$\mathbb{P}(T_r < T_R) = \frac{\log R - \log |x|}{\log R - \log r}.$$

(b) Suppose now that $B_0 = 0$. Let $0 < r_1 < r_2$, and define stopping times $\tau_0 = 0$ and

$$\tau_{2k+i} = \inf\{t \geqslant \tau_{2k+i-1} : |B_t| = r_i\} \quad \text{for integers } k \geqslant 0 \quad \text{and} \quad i \in \{1, 2\}.$$

For any $R > r_2$, let

$$N(R) = \sup\left\{k \in \mathbb{N} : \sup_{0 \leqslant t \leqslant \tau_{2k}} |B_t| < R\right\}.$$

Show that

$$\frac{N(R)}{\log R} \to Y \text{ in distribution as } R \to \infty,$$

where Y is an exponential random variable of parameter $\log(r_2/r_1)$.

[Throughout this question you may use results from the course as long as they are stated clearly.]

4 (a) State *Donsker's invariance principle* in one dimension.

(b) Let B be a one dimensional Brownian motion. Let $\mu > 0$, and consider

$$\widetilde{B}_t = B_t - \mu t \quad \text{for} \quad t \geqslant 0 \quad \text{and} \quad S = \sup_{s \geqslant 0} \widetilde{B}_s.$$

(i) Show that $\lim_{t\to\infty} \widetilde{B}_t = -\infty$ almost surely.

(ii) Show that for $x, y > 0$,

$$\mathbb{P}(S \geqslant x + y) = \mathbb{P}(S \geqslant x)\,\mathbb{P}(S \geqslant y)\,.$$

(iii) Let $T_x = \inf\{t \geqslant 0 : B_t = x\}$. It is known that for every $\lambda > 0$ and $x > 0$, we have $\mathbb{E}\big[e^{-\lambda T_x}\big] = e^{-\sqrt{2\lambda}x}$. You may use this fact without proof. Deduce that S has the exponential distribution with parameter 2μ.

[Throughout this question you may use without proof that a continuous random variable with the property of part (ii) has the exponential distribution. You may use standard properties of Brownian motion without proof.]

5 Let $(B_t)_{t \geqslant 0}$ be a two dimensional Brownian motion. For every $a > 0$, write $\tau_{a,r} = \inf\{t \geqslant 0 : |B_t - a| = r\}$. For every $x \in \mathbb{R}^2$ and $r > 0$, we denote by $\nu_{x,r}$ the uniform measure on $\{y : |y - x| = r\}$.

Let $a = (-1, 0)$ and $0 < r_1 < r_2$ be such that $B(a, r_1) \subset B(0, r_2)$. The goal of this question is to show that if $B_0 \sim \nu_{0,r_2}$, then $B_{\tau_{a,r_1}} \sim \nu_{a,r_1}$, i.e. starting the Brownian motion uniformly from $\{y : |y| = r_2\}$, the first hitting point of $B(a, r_1)$ is uniformly distributed.

(a) Let $0 < r < R$. Show that if $B_0 \sim \nu_{0,R}$, then

$$B_{\tau_{0,r}} \sim \nu_{0,r}.$$

(b) Let $R > 2r_2$, and let x, y be such that $|x| = |y| = R$. Let L be the line passing through the origin perpendicular to the line passing through x and y and $T_L = \inf\{t \geqslant 0 : B_t \in L\}$. Show that if $A \subseteq \{z : |z - a| \leqslant r_1\}$ is a Borel set, then

$$|\mathbb{P}_x\big(B_{\tau_{a,r_1}} \in A\big) - \mathbb{P}_y\big(B_{\tau_{a,r_1}} \in A\big)| \leqslant \mathbb{P}_x(\tau_{a,r_1} < T_L)\,.$$

[*Hint: You may wish to couple the Brownian motions starting from x and y as follows: let ϕ be the reflection with respect to the line L. Define $B'_t = \phi(B_t)$ for $t \leqslant T_L$ and $B'_t = B_t$ for $t > T_L$.*]

(c) Show that for every $\varepsilon > 0$, there exists $R > 0$ sufficiently large so that for every Borel set $A \subseteq \{z : |z - a| \leqslant r_1\}$,

$$|\mathbb{P}_{\nu_{a,R}}\big(B_{\tau_{a,1}} \in A\big) - \mathbb{P}_{\nu_{0,R}}\big(B_{\tau_{a,1}} \in A\big)| \leqslant \varepsilon.$$

(d) Conclude that if $B_0 \sim \nu_{0,r_2}$, then $B_{\tau_{a,r_1}} \sim \nu_{a,r_1}$.

[Throughout this question you may use results from the course provided they are stated clearly.]

6

(a) State the *martingale convergence theorem* for a uniformly integrable martingale.

(b) Let X be a discrete time uniformly integrable martingale. Let $S \leqslant T$ be stopping times. Show that

$$\mathbb{E}[X_T] = \mathbb{E}[X_S]\,.$$

(c) Let X be a discrete time martingale with values in $\mathbb{Z}$. Suppose that $X_0 = x > 0$ and for every n, we have $|X_{n+1} - X_n| \leqslant 1$ and $\mathbb{P}(|X_{n+1} - X_n| = 1 \mid \sigma(X_0, \ldots, X_n)) \geqslant 1/2$. For every $a \in \mathbb{Z}$, set

$$T_a = \inf\{n \geqslant 0 : X_n = a\}.$$

For every $y > x$, show that

$$\mathbb{P}(T_y < T_0) = \frac{x}{y}.$$

END OF PAPER