UNIVERSITY OF CAMBRIDGE

MAMA/152, NST3AS/152, MAAS/152

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2024 1:30 pm to 3:30 pm

PAPER 152

TORIC VARIETIES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
Both questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) When does a strongly convex cone $\sigma \subset N_{\mathbb{R}}$ define a smooth toric variety U_σ? State, without proof, necessary and sufficient conditions on σ.

(b) State the Orbit-Cone correspondence theorem.

(c) Let X_Σ be the toric variety associated to a fan Σ with $|\Sigma| \subset N_{\mathbb{R}}$. Let

$$X_\Sigma^{\text{sing}} = \{x \in X_\Sigma | x \text{ is a singular (closed) point.}\}$$

Prove that X_Σ^{sing} is a union of orbits of T_N, using the parts above or otherwise. You may use any of the results from lectures without proof.

(d) Let $\sigma = \text{Cone}(3e_1 - 2e_2, e_2) \subset \mathbb{R}^2$, and let $N = \mathbb{Z}^2$ be the standard lattice in $\mathbb{R}^2$ with generators e_1, e_2. Let M be the dual lattice on N with dual generators e_1^*, e_2^*.

(i) Find the coordinate ring of U_σ as a subring of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, where $x = \chi^{e_1^*}$, $y = \chi^{e_2^*}$.

(ii) Show that U_σ is a singular toric variety and find a fan Σ which induces a toric resolution of singularities

$$p : X_\Sigma \to U_\sigma.$$

(iii) Let $\tau_1 = \text{Cone}(3e_1 - 2e_2)$ and $\tau_2 = \text{Cone}(e_2)$. For $i = 1, 2$ let D_i be the Weil divisor on U_σ corresponding to the orbit-closure $V(\tau_i)$. Compute the class group $\text{Cl}(U_\sigma)$ in terms of generators and relations.

2

(a) Let $N_{\mathbb{R}} = \mathbb{R}^2$ with standard basis e_1, e_2. Let Σ be the fan with support in $N_{\mathbb{R}}$ with rays $u_0 = \mathrm{Cone}(e_1 + e_2)$, $u_1 = \mathrm{Cone}(e_1)$, $u_2 = \mathrm{Cone}(e_2)$ and $u_3 = \mathrm{Cone}(-e_1)$ and 2-dimensional cones $\mathrm{Cone}(u_0, u_1)$, $\mathrm{Cone}(u_0, u_2)$, $\mathrm{Cone}(u_2, u_3)$.

(i) Show that X_Σ is the blowup of $\mathbb{P}^1 \times \mathbb{C}$ at a torus-fixed point.

(ii) Let N be the lattice in $N_{\mathbb{R}}$ with $\mathbb{Z}$-basis e_1, e_2. Let N' be a rank 1 lattice generated by e'. Show that the lattice morphism $\phi : N \to N'$ given by $\phi(e_1) = 0$, $\phi(e_2) = e'$ induces a toric morphism

$$f : X_\Sigma \to \mathbb{C}.$$

(iii) Show that $f^{-1}(0)$, i.e. the fiber at $0 \in \mathbb{C}$, consists of two copies of $\mathbb{P}^1$ meeting at one point.

(iv) Identify the toric variety $X_{\Sigma_0} := X_\Sigma \setminus f^{-1}(0)$. That is, find its fan Σ_0 in $N_{\mathbb{R}}$ and identify it with a known variety.

(v) Compute the fibers $f^{-1}(x)$ for all closed points $x \in \mathbb{C}$.

(b) Let $\sigma = \mathrm{Cone}(e_1, e_2, e_3) \subset N_{\mathbb{R}} = \mathbb{R}^3$, where e_1, e_2, e_3 are the standard generators of $N = \mathbb{Z}^3$. Let $N'_{\mathbb{R}} \cong \mathbb{R}^2$ with basis f_1, f_2 be the support of the complete fan Σ with rays f_1, f_2, $-(f_1 + f_2)$.

(i) Sketch the fans and identify the varieties U_σ and X_Σ.

(ii) Let $N = \mathbb{Z}^3$ and $N' = \mathbb{Z}^2$ be the standard lattices in the vector spaces $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ respectively. Consider the lattice morphism $\widehat{\varphi} : N \to N'$ defined by

$$\widehat{\varphi} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Show that $\widehat{\varphi}$ is **not** a morphism of fans from σ to Σ. Find the rational map $f : U_\sigma \dashrightarrow X_\Sigma$ defined by $\widehat{\varphi}$. What is the indeterminacy locus of f?

(iii) Give a refinement $\widetilde{\Sigma}$ of the cone σ which resolves the indeterminacy locus of f. That is, show that for your choice of $\widetilde{\Sigma}$ the lattice morphism $\widehat{\varphi}$ induces a morphism of fans from $\widetilde{\Sigma}$ to Σ. Identify the toric variety $X_{\widetilde{\Sigma}}$ and the induced toric morphism $g : X_{\widetilde{\Sigma}} \to X_\Sigma$.

END OF PAPER