UNIVERSITY OF CAMBRIDGE

MAMA/151, NST3AS/151, MAAS/151

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 1:30 pm to 3:30 pm

PAPER 151

GROUP COHOMOLOGY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let G be a finite group and let M be a $\mathbb{Z}G$-module. Define the cohomology group $H^n(G, M)$. [You may assume independence of choice of resolution.]

Let M_1 and M_2 be $\mathbb{Z}G$-modules. Show that $H^n(G, M_1 \bigoplus M_2) = H^n(G, M_1) \bigoplus H^n(G, M_2)$.

Let K be a subgroup of G.

For a $\mathbb{Z}K$-module X, let Y be the coinduced module $\mathrm{Hom}_K(\mathbb{Z}G, X)$ with G-action given by $(gf)(r) = f(rg)$, where $f \in \mathrm{Hom}_K(\mathbb{Z}G, X)$ and $r \in \mathbb{Z}G$.

Prove Shapiro's Lemma, that $H^n(K, X) = H^n(G, Y)$.

Consider $\mathbb{Z}G$ as a $\mathbb{Z}G$-module N via conjugation $g.x = gxg^{-1}$ for $g \in G$ and $x \in \mathbb{Z}G$.

Show that $H^n(G, N) = \bigoplus H^n(C_G(g_i), \mathbb{Z})$ where the direct sum is taken over a set of conjugacy class representatives g_i and $C_G(g_i)$ is the centraliser of g_i in G.

Now let $G = S_3$, and calculate $H^n(S_3, N)$ for $n = 1$ and for $n = 2$. [You may assume that $H^2(S_3, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ but you should prove any other results you use.]

2

Define the Schur multiplier $M(G)$ of a group G.

State Hopf's formula for $M(G)$ in terms of a presentation of the group G.

By writing down (without proof) a partial free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ arising from the group presentation, prove Hopf's formula.

Calculate the Schur multiplier of the abelian group $C_2 \times C_4 \times C_6$.

3

Let R be a (not necessarily commutative) ring, and let I be a (2-sided) ideal of R such that $I^2 = 0$.

Show that $1 + I = \{1 + r : r \in I\}$ forms an abelian group under multiplication and that it is isomorphic to the additive group of I.

Suppose you are given an isomorphism between R/I and $\mathbb{Z}G$ for a group G.

Show that left multiplication in R induces a left $\mathbb{Z}G$-module structure on I, and similarly on the right. Deduce that there is a conjugation action of G on I, and on $1 + I$ such that I and $1 + I$ are isomorphic as $\mathbb{Z}G$-modules.

Show that multiplication in R yields an extension of groups

$$1 \to 1 + I \to U \to G \to 1$$

for some subgroup U of the multiplicative units of R. Explain how to obtain an element x of $H^2(G, 1 + I)$ corresponding to this extension.

Suppose that you also have a ring R_1 with ideal I_1 such that $I_1^2 = 0$, and an isomorphism $R_1/I_1 \to \mathbb{Z}G$. Similarly obtain an extension $1 \to 1 + I_1 \to U_1 \to G \to 1$ for a subgroup U_1 of units of R_1, and an element x_1 of $H^2(G, 1 + I_1)$.

Suppose that θ is a $\mathbb{Z}G$-module isomorphism $1 + I_1 \to 1 + I$ and ϕ is the induced isomorphism $H^2(G, 1 + I_1) \to H^2(G, 1 + I)$.

Suppose that R and R_1 are isomorphic as rings. Is $\phi(x_1)$ necessarily equal to x? Justify your answer.

4

Let G be a group with normal subgroup H. Let M be a $\mathbb{Z}G$-module.

State the associated five term exact sequence of cohomology, carefully describing the maps involved.

Let K be a finite non-abelian simple group which is isomorphic to F/R where F is a free group of rank n and R is generated as a normal subgroup of F by n elements. Show that $H^2(K, \mathbb{Z}) = 0$ (where the action on $\mathbb{Z}$ is trivial).

END OF PAPER