UNIVERSITY OF CAMBRIDGE

MAMA/146, NST3AS/146, MAAS/146

MAT3

MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 1:30 pm to 3:30 pm

PAPER 146

SYMPLECTIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State and prove the "two-out-of-three" property for $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{Sp}(2n)$, and $\mathrm{O}(2n)$.

(b) Write down a homeomorphism from the Lagrangian Grassmanian $\mathrm{LGr}(\mathbb{R}^{2n})$ to a coset space G/H of two matrix groups. Illustrate your map in the $n = 1$ case.

(c) By considering a suitable map to $\mathrm{LGr}(\mathbb{R}^2)$ or otherwise, show that for any $n \geqslant 1$ the group $H^1(\mathrm{LGr}(\mathbb{R}^{2n}); \mathbb{Z})$ is non-zero.

(d) Prove carefully that the fibre bundle $\mathcal{LG}r(TS^2) \to S^2$ is not smoothly trivial for any choice of symplectic structure on S^2.

(e) For any $n \geqslant 1$, exhibit a compact symplectic manifold (M, ω) of dimension $2n$ such that the fibre bundle $\mathcal{LG}r(TM) \to M$ is smoothly trivial.

2 State the Weinstein neighbourhood theorem. Give examples (with proof) of

(i) a smooth, *orientable* manifold L such for any Lagrangian embedding of L into a symplectic manifold (M, ω), its image is non-displaceable under smooth isotopy. Give also an example of such a Lagrangian embedding with M compact.

(ii) a Lagrangian L in a compact symplectic manifold M which is displaceable under smooth isotopy, but not symplectic isotopy;

(iii) a Lagrangian L in a compact symplectic manifold M which is displaceable under arbitrarily small symplectic isotopy (that is, for all $\epsilon > 0$, there exists some symplectic isotopy ϕ_t such that $\phi_1(L) \cap L = 0$ and $\|\phi_t\|_{C^\infty} \leqslant \epsilon$ for all t) but not displaceable under *any* Hamiltonian isotopy;

(iv) a Lagrangian in a compact symplectic manifold M which is displaceable under Hamiltonian isotopy.

3 State and prove Moser's theorem on families of symplectic forms on compact manifolds.

Let $d \geqslant 1$ and $n \geqslant 2$. Briefly explain how to deduce that any pair X and X' of smooth degree d hypersurfaces in $\mathbb{P}^n$ are symplectomorphic. Show that for any such hypersurface X in $\mathbb{P}^n$, the symplectomorphism group $\mathrm{Symp}(X, \omega_{FS}|_X)$ has a subgroup isomorphic to

$$(\mathbb{Z}/d\mathbb{Z})^{n+1}/(1, \cdots, 1) \cong (\mathbb{Z}/d\mathbb{Z})^n.$$

4 (a) State the neighbourhood theorem for symplectic submanifolds.

(b) Let (X^4, ω) be a symplectic manifold and C be a smooth genus zero symplectic submanifold of X with $C \cdot C = -4$. Explain how to symplectically blow-down C.

(c) Let $E(1)$ be the complex surface constructed by blowing up the base points of a general pencil of cubics. Write a holomorphic map $\pi : E(1) \to \mathbb{P}^1$ with connected fibres and exhibit a holomorphic section of that map. Let $C \subset E(1)$ be a smooth, connected complex curve. Suppose that $C \cdot C = k$ and that $\pi|_C : C \to \mathbb{P}^1$ has degree d. What is the genus of C? Justify your answer.

END OF PAPER