MAMA/137

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 1:30 pm to 4:30 pm

PAPER 137

MODULAR FORMS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let k be an integer. Write down a formula for the dimension of the space $S_k(\Gamma(1))$ of cuspidal modular forms of weight k and level $\Gamma(1)$.

(b) Consider the function $E_2 : \mathfrak{h} \to \mathbb{C}$ defined by the formula

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$$

where $q = e^{2\pi i \tau}$ and $\sigma(n) = \sum_{d|n} d$. Using the formula $\tau^{-2} E_2(-1/\tau) = E_2(\tau) + \frac{6}{\pi i \tau}$, prove the formula

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $\Delta \in S_{12}(\Gamma(1))$ is the unique cuspidal modular form of weight 12 satisfying $\Delta = q + O(q^2)$ as $q \to 0$. [You may assume that the group $\Gamma(1)$ is generated by the matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.]

(c) Show that $L(\Delta, s) \neq 0$ for all real values of s with $s > 0$. [You may assume that the Dirichlet series defining $L(\Delta, s)$ is absolutely convergent whenever $\mathrm{Re}(s)$ is sufficiently large, and that $L(\Delta, s)$ admits an analytic continuation to $\mathbb{C}$.]

2 (a) Define

$$\mathcal{F} = \{\tau \in \mathfrak{h} \mid |\tau| \geqslant 1, \mathrm{Re}(\tau) \in [-1/2, 1/2]\}.$$

Prove that every element of $\mathfrak{h}$ is $\Gamma(1)$-conjugate to an element of $\mathcal{F}$.

(b) Let $k \geqslant 0$ be an integer. Define the Petersson inner product $\langle \cdot, \cdot \rangle$ on the space $S_k(\Gamma(1))$ of cuspidal modular forms of weight k and level $\Gamma(1)$, and show that it converges.

(c) Let $k, l \geqslant 0$ be even integers, and let $f = \sum_{n \geqslant 1} a_n q^n \in S_k(\Gamma(1))$, $g = \sum_{n \geqslant 1} b_n q^n \in S_l(\Gamma(1))$. Show that the Dirichlet series $L(f, g, s) = \sum_{n \geqslant 1} a_n \bar{b}_n n^{-s}$ converges absolutely when $\mathrm{Re}(s) > 1 + (k + l)/2$. [You may assume there is a constant $C > 0$ such that $|a_n| \leqslant C n^{k/2}$ and $|b_n| \leqslant C n^{l/2}$ for all $n \geqslant 1$.]

(d) With the assumptions of (c), suppose further that $k \geqslant l + 6$. Prove the identity

$$\langle f, g G_{k-l} \rangle = \frac{2\zeta(k - l)\Gamma(k - 1)}{(4\pi)^{k-1}} L(f, g, k - 1),$$

where $G_{k-l} = \sum_{(m,n) \in \mathbb{Z}^2 - 0} (m\tau + n)^{-(k-l)}$ is the usual weight $(k - l)$ Eisenstein series. [You may use any result from lectures provided you state it precisely.]

3 (a) Let k be an integer, and let $\Gamma \leqslant \Gamma(1)$ be a congruence subgroup. Define what it means for a function $f : \mathfrak{h} \to \mathbb{C}$ to be a modular form of weight k and level Γ.

(b) Let f be a non-zero modular function of weight k and level $\Gamma(1)$. Write down a formula for the number of zeroes and poles of f in $\Gamma(1)\backslash\mathfrak{h}$.

(c) Let $\theta : \mathfrak{h} \to \mathbb{C}$ be defined by $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$. Show that if $k \in 8\mathbb{N}$ then θ^k is a modular form of weight $k/2$ and level Γ, where $\Gamma = \Gamma(2) \cup \Gamma(2)S$. [You may use the fact that the group Γ is generated by $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. You may use the Poisson summation formula, provided you state it precisely.]

(d) Show that $\theta(\tau) \neq 0$ for all $\tau \in \mathfrak{h}$.

4 (a) (i) Let $k \in \mathbb{Z}$, let $n \in \mathbb{N}$, and let $\mathcal{L}$ denote the set of lattices $\Lambda \leqslant \mathbb{C}$. Define the space V_k of functions $F : \mathcal{L} \to \mathbb{C}$ of weight k and the n^{th} Hecke operator $T_n : V_k \to V_k$.

(ii) Explain how V_k may be identified with the space W_k of functions $f : \mathfrak{h} \to \mathbb{C}$ that are invariant under the weight k action of $\Gamma(1)$, and how this can be used to define the n^{th} Hecke operator $T_n : W_k \to W_k$.

(b) Let $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. The non-holomorphic Eisenstein series $G(\tau, s)$ of parameter s is defined for $\tau \in \mathfrak{h}$ by

$$G(\tau, s) = \sum_{(m,n)\in\mathbb{Z}^2-0} \frac{\mathrm{Im}(\tau)^s}{|m\tau + n|^{2s}}.$$

Prove that $G(\tau, s)$ converges absolutely for all $\tau \in \mathfrak{h}$ and defines an element of W_0.

(c) Let p be a prime. Show that $G(\tau, s)$ is an eigenvector for the Hecke operator T_p, and compute the eigenvalue.

END OF PAPER