UNIVERSITY OF CAMBRIDGE

MAMA/136, NST3AS/136, MAAS/136

MAT3

MATHEMATICAL TRIPOS Part III

Monday 3 June 2024 9:00 pm to 12:00 pm

PAPER 136

LOCAL FIELDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove a version of Hensel's Lemma and use it to show that there is an isomorphism

$$\mathbb{Q}_3^\times/(\mathbb{Q}_3^\times)^3 \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

(b) Consider the equation

$$X^2 = a^4 + 4 \in \mathbb{Q}_2[X]$$

where $a \in \mathbb{Q}_2$. Show that for $|a|_2$ sufficiently *large*, this equation has a solution in $\mathbb{Q}_2$.

2 Let K be a local field and L/K a finite separable extension.

(a) Define what it means for L/K to be unramified, totally ramified and tamely ramified. Show that for any extension L/K, there is a subextension K_0/K such that K_0/K is unramified and L/K_0 is totally ramified.

(b) Now let L/K be Galois with Galois group G. Define the higher ramification groups $G_s(L/K)$ for $s \in \mathbb{Z}_{\geqslant -1}$. Assume that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha \in \mathcal{O}_K$ with minimal polynomial $f(X) \in \mathcal{O}_K[X]$, and let v_L denote the normalized valuation on L. Show that $G_s(L/K) = \{\sigma \in G | v_L(\sigma(\alpha) - \alpha) \geqslant s + 1\}$, and we have

$$v_L(f'(\alpha)) = \sum_{1 \neq \sigma \in G} v_L(\sigma(\alpha) - \alpha) = \sum_{s \in \mathbb{Z}_{\geqslant 0}} (|G_s| - 1).$$

Deduce that L/K is unramified if and only if $f'(\alpha) \in \mathcal{O}_L^\times$ and that L/K is tamely ramified if and only $v_L(f'(\alpha)) = e_{L/K} - 1$. [*You may use standard facts about higher ramification groups without proof.*]

3 Let K be a finite extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$ and $\pi \in \mathcal{O}_K$ a uniformizer. Let $\overline{K}$ be an algebraic closure of K and $\overline{\mathfrak{m}} \subset \mathcal{O}_{\overline{K}}$ the maximal ideal.

(a) Let f be a Lubin–Tate series for π. Define the Lubin–Tate formal group law $\mathcal{F}_f$ associated to f, and show that if g is another Lubin–Tate series for π, there is an isomorphism $\mathcal{F}_f \cong \mathcal{F}_g$ of formal $\mathcal{O}_K$-modules.

[*You may assume the key lemma on existence and uniqueness of power series associated to Lubin–Tate series provided it is stated clearly.*]

(b) For $n \geqslant 1$, define the π^n-torsion points $\mu_{f,n} \subset \overline{\mathfrak{m}}$ associated to K and show that $\mu_{f,n}$ is a free module of rank one over $\mathcal{O}_K/\pi^n\mathcal{O}_K$.

(c) By considering appropriate Lubin–Tate series for p, show that

$$\mathbb{Q}_p(\sqrt[p-1]{-p}) = \mathbb{Q}_p(\zeta_p).$$

4 Let K be a complete discretely valued field with perfect residue field k of characteristic $p > 0$, and let $\pi \in \mathcal{O}_K$ be a uniformizer.

(a) Show that there exists a unique multiplicative map

$$[.] : k \to \mathcal{O}_K$$

such that $[a] \equiv a \mod \pi$ for all $a \in k$. Hence show that every $x \in \mathcal{O}_K$ can be written as a power series $x = \sum_{i=0}^{\infty} [a_i]\pi^i$, $a_i \in k$.

(b) Show that if $\mathrm{char} K = 0$ and k is finite, there is an isomorphism $(1 + \pi^r \mathcal{O}_K) \cong (\mathcal{O}_K, +)$ for sufficiently large r.

Let $K = \mathbb{Q}_3(\zeta_3)$. Show that there is an isomorphism $\mathcal{O}_K^\times \cong \mu_6 \times \mathcal{O}_K$, where μ_6 is the group of 6^{th} roots of unity.

5

Let K be an algebraic number field.

(a) State Ostrowski's Theorem. For p a prime, show that the absolute values on K extending $|.|_p$ are (up to equivalence) given by $|.|_{\mathfrak{p}}$ for $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$ with $\mathfrak{p} \cap \mathbb{Z} = (p)$.

Show that there is a natural isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}.$$

(b) Let $K = \mathbb{Q}(\sqrt[3]{2})$. Show that there are two prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ in $\mathcal{O}_K$ dividing the prime 5. Are either of these ramified?

END OF PAPER