MAMA/134, NST3AS/134, MAAS/134

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 9:00 am to 11:00 am

PAPER 134

CUBULATING SPACES AND GROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a. Define the right angled Artin group associated to a graph Γ and its corresponding Salvetti complex.

b. Give sufficient conditions, in terms of the defining graph Γ, for the right angled Artin group $G(\Gamma)$ to be a free product or a direct product. Describe what this means for the structure of the corresponding Salvetti complexes.

c. Prove that the link of the vertex in the Salvetti complex of a right angled Artin group is a cycle if and only if Γ is an edge.

d. Show that a surface of genus 2 cannot be homeomorphic to a hyperplane in the Salvetti complex of a right angled Artin group (you may quote, without proof, any result from the example sheets).

e. Prove that if $G(\Gamma)$ is a right angled Artin group and Γ' is a full subgraph of Γ, then $G(\Gamma')$ is isomorphic to a subgroup of $G(\Gamma)$.

2 Let G be a group, $(S, \mathcal{W})$ be a wallspace on which G acts, and C be the dual CAT(0) cube complex to $(S, \mathcal{W})$.

a. Define what it means for G to be *cubulated* and *cocompactly cubulated.*

b. State a criterion guaranteeing that the action of G on C is metrically proper.

c. State and prove a sufficient and necessary condition for the action on C to be cocompact.

d. The Coxeter group
$$W = \langle r, s, t | s^2, t^2, r^2, (st)^3, (rs)^3, (rt)^3 \rangle$$
is isomorphic to the group of isometries of the Euclidean plane $\mathbb{E}^2$ tessellated by equilateral triangles, generated by reflections on 3 pairwise non-parallel lines, as partially shown in Figure 1. By defining a wallspace structure on $\mathbb{E}^2$ where the walls are given by the lines in the tessellation, we obtain an action of W on a CAT(0) cube complex C. Describe C and prove that the action of W on C is **not** cocompact.

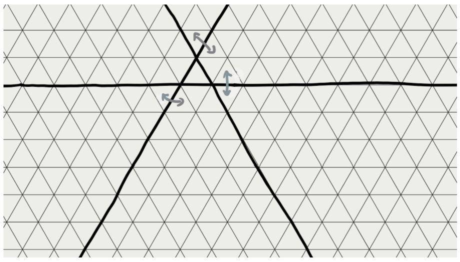


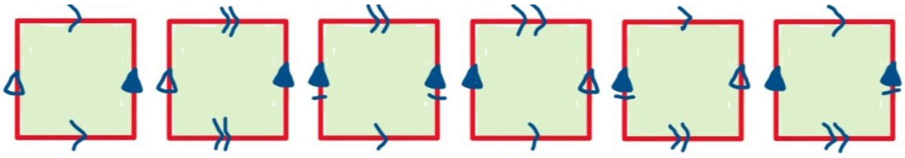
Figure 1: W is generated by reflection on three pairwise-crossing lines; a portion of the tessellation of $\mathbb{E}^2$ is shown.

3

a. Define what it means for a group G to be residually finite. Prove that free groups are residually finite.

b. Show that a subcomplex of a product of two graphs is special.

c. Determine whether the cube complex below is special or not. Prove your assertion.



d. Recall that a group G is *simple* if the only normal subgroups of G are the trivial subgroup and G itself. Prove that if a compact, connected, non-positively curved cube complex X is special, and $\pi_1 X$ is not finite, then $\pi_1 X$ cannot be simple.

END OF PAPER