UNIVERSITY OF CAMBRIDGE

MAMA/133, NST3AS/133, MAAS/133

MAT3

# MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 1:30 pm to 4:30 pm

## PAPER 133

## GEOMETRIC GROUP THEORY

### Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** In this question, $x^y$ denotes the conjugate $y^{-1}xy$. Consider the group

$$G = \langle a, t \mid (a^t)a(a^t)^{-1}a^{-2} \rangle \,.$$

(a) Let $f : G \to Q$ be a surjective homomorphism to a finite group. By considering the order of $f(a)$, prove that $Q$ is cyclic. [*You may use the following consequence of Fermat's little theorem without proof: if $n$ is a positive integer and $n$ divides $2^n - 1$, then $n = 1$.*]

(b) Prove that $G$ is infinite and non-abelian.

(c) Prove that the inclusion of $\langle a \rangle$ into $G$ is not a quasi-isometric embedding.

**2** Consider the group $SL_2(\mathbb{Z})$ of 2-by-2 matrices of determinant 1 with integer coefficients. You may assume that $SL_2(\mathbb{Z})$ can be written as an amalgamated free product $\langle A \rangle *_{\langle A \rangle \cap \langle B \rangle} \langle B \rangle$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Draw a picture of the Bass–Serre tree $T$ corresponding to the given decomposition of $SL_2(\mathbb{Z})$.

(b) Prove that any isometry of $T$ either fixes a point or translates a line. Deduce that every element of $SL_2(\mathbb{Z})$ of finite order is conjugate to either a power of $A$ or a power of $B$.

(c) Prove that $SL_2(\mathbb{Z})$ has a free subgroup of finite index. [*Hint: consider the quotient of $SL_2(\mathbb{Z})$ induced by taking residues modulo 3.*]

**3** Consider a homomorphism of groups $f : G \to H$, with $H$ finitely generated.

(a) Suppose that $f$ is injective and $|H : f(G)| < \infty$. Prove that $G$ is finitely generated and quasi-isometric to $H$.

(b) Suppose instead that $f$ is surjective and $\ker f$ is finite. Prove that $G$ is finitely generated and quasi-isometric to $H$.

(c) Show that the group

$$\Gamma = \langle a, b, k \mid a^4, b^4, (abk)^4, k^3, aka^{-1}k, bkb^{-1}k \rangle$$

is quasi-isometric to the hyperbolic plane $\mathbb{H}^2$.

**4** Let $X$ be a $\delta$-hyperbolic metric space.

(a) Let $Q \subseteq X$ be a geodesic quadrilateral. Prove that each side is contained in the closed $2\delta$-neighbourhood of the remaining three sides.

(b) Consider a subspace $Y \subseteq X$ that is *convex*, meaning that any geodesic with both endpoints in $Y$ is contained in $Y$. Fix a point $x \in X$. Suppose that $y_1, y_2 \in Y$ both have the property that $d(x, y_i) \leqslant d(x, y)$ for all $y \in Y$. Prove that $d(y_1, y_2) \leqslant 4\delta$.

(c) Consider convex subspaces $Y, Z \subseteq X$, such that $d(y, z) > 2\delta$ for all $y \in Y$ and $z \in Z$. Let $z_1, z_2$ be points in $Z$, and suppose $y_1, y_2 \in Y$ have the property that $d(y_i, z_i) \leqslant d(y, z_i)$ for all $y \in Y$ and $i = 1, 2$. Prove that $d(y_1, y_2) \leqslant 8\delta$.

## END OF PAPER